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GENERAL THEORY OF

INTEGRAL FUNCTIONS

LECTURES

ON THE

GENERAL THEORY OF

INTEGRAL FUNCTIONS

BY

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PREFACE

These lectures have a twofold interest.

Delivered in French by a prominent French scholar to an English speaking audience consisting mainly of Honours students of whose curriculum the subject matter formed an integral part, they are noteworthy as constituting the permanent record of a successful experiment, carried out at the University College of Wales during the period of reorganisation of its Pure Mathematical Department.

But their scientific interest is far greater than this might lead us to suppose. The lectures give us, in the form of a number of elegant and illuminating theorems, the latest word of mathematical science on the subject of Integral Functions. And they do more. They descend to details, they take us into the workshop of the working mathematician, they explain to us the nature of his tools, and shew us the way to use them; while, at the same time, by the absence of any attempt to conceal the imperfections of the edifice so far constructed, they indicate to us the work still waiting to be done, they inspire us with the desire and furnish us with the means of completing it ourselves.

The book will not be found difficult by an earnest student. He may hope to master it without any elaborate preliminary preparation. All that is needed is that he should possess some familiarity with the more immediate consequences of the fundamental and far reaching theorem due to the genius of Cauchy and known by his name and that he should be at home

with the concept of number and its extensions such as form to day the foundations of Analysis.

For the philosophic mathematician the subject is particularly instructive, shewing, as it does, the power of a single fundamental idea, that of generalisation, germinating in many minds of widely different types of training, and applied to a particular mathematical concept, that of the polynomial.

A problem once stated is, as we all know, half solved; and the rapidity with which one step in our theory has followed another will be seen by a glance at the bibliography at the end of the volume. In the same connexion the appendices may specially be cited. They contain among other things the complete solution of questions connected with the inverse function of an integral function, questions raised and only partially answered by Hurwitz in two Notes in the Comptes Rendus. They contain also an illustration of the growing influence of the Theory of Sets of Points, not only on the language of the Theory of Integral Functions, but also on its subject matter; and we are led to ask whether in the near future the whole trend of research may not here also be in this direction.

W.-H. Young.

The Athenaeum,
London.
June, 1923.

AVERTISSEMENT

Ce livre reproduit les leçons que j'ai eu l'honneur de faire aux élèves de M. Young à l'University College of Wales à Aberystwyth en février et mars 1922. Qu'il me soit permis de renouveler ici l'expression de ma vive reconnaissance au professeur Young qui m'a invité à faire ces leçons, qui m'a donné les moyens de les publier et qui a bien voulu les présenter au public. Si cet ouvrage rend quelques services aux étudiants, que leur gratitude aille d'abord à M. Young qui en a été le promoteur

Mes leçons étaient faites en français ; un jeune mathématicien de Cambridge, M. Collingwood, me seconda en donnant des explications complémentaires à mes auditeurs et accepta de remplir le rôle ingrat de traducteur. Je l'en remercie bien vivement.

Le nombre de mes leçons était limité et je me proposais de permettre à des auditeurs possédant les éléments de la théorie des fonctions analytiques de me suivre sans difficulté, j'ai donc pris la théorie à son point de départ et ai dû par contre sacrifier des développements intéressants. Sur un point cependant je me suis départi de la règle que je m'étais fixée de ne m'appuyer que sur des propositions élémentaires : au chapitre VI j'ai admis les propriétés de la fonction modulaire elliptique $J(\omega)$, le lecteur en trouvera un exposé très clair dans un Mémoire de Hurwitz cité dans la bibliographie. Le texte des six chapitres de l'ouvrage, qui donne sans modifications mes leçons, a été rédigé en octobre 1921. M. Collingwood a bien

voulu y ajouter les appendices A et B qui complètent certaines questions traitées dans le texte, et l'appendice D où il établit, d'après les travaux de M. Iversen, la correspondance entre les points singuliers non algébriques de la fonction inverse d'une fonction entière et les valeurs asymptotiques de cette fonction. Je signale à ce sujet que la définition des points singuliers adoptée dans cet appendice est celle donnée par M. Bieberbach dans l'encyclopédie (article II C 4 de l'édition allemande). Dans l'appendice C, Miss C. Young expose quelques résultats relatifs à la distribution des zéros des fonctions d'ordre fini entier. Grâce à ces compléments, l'ouvrage, bien qu'il n'ait pas la prétention d'être une encyclopédie, renferme l'essentiel de la théorie. La lecture de la table des matières et des petites introductions de chaque chapitre mettra rapidement le lecteur au courant des questions traitées, ce qui me dispense d'en parler ici. Une bibliographie importante termine l'ouvrage; dans le texte, chaque chapitre est suivi d'une liste indiquant ceux des Mémoires indiqués dans la bibliographie qui ont été utilisés pour la rédaction des divers paragraphes. Les épreuves ont été revues par Miss C. Young et par M. Collingwood; je les remercie de cette aimable collaboration.

Le livre, tel qu'il est conçu, permettra, je crois, aux jeunes mathématiciens de langue anglaise de se familiariser avec des questions qui ont été peu travaillées jusqu'ici par leurs compatriotes. Je pense qu'ils apercevront sans peine, notamment en comparant les résultats des chapitres III et VI, les points qui appellent les premières recherches et j'espère que, grâce à ces énergies nouvelles, la théorie sera complétée et rajeunie. Alors ce livre n'aura plus guère d'intérêt, mais le but que je me suis proposé en l'écrivant sera pleinement atteint.

Strasbourg, le 20 avril 1923.

Georges VALIRON.

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THE GENERAL THEORY OF INTEGRAL FUNCTIONS

CHAPTER I

Preliminary Notions.

Introduction. — The origin of the general theory of integral functions, that is to say of functions which are regular throughout the finite portion of the plane of the complex variable, is to be found in the work of Weierstrass. He shewed that the fundamental theorem concerning the factorisation of a polynomial can be extended to cover the case of such functions, and that in the neighbourhood of an isolated essential singularity the value of a uniform function $F(z)$ is indeterminate. These two theorems have been the starting point of all subsequent research.

Weierstrass himself did not complete his second theorem. This was done in 1879 by Picard who proved that in the neighbourhood of an isolated essential singularity a uniform function actually assumes every value with only one possible exception. Much important work, the earliest of which was due to Borel, has been done in connection with *Picard's theorem;* and the consequent introduction of new methods has resulted in much light being thrown on obscure points in the theory of analytic functions.

The notion of the *genus* of a Weierstrassian product was introduced and its importance first recognised by Laguerre, but it was not until after the work of Poincaré and Hadamard had been done that any substantial advance was made in this direction. Here also Borel has enriched the theory with new ideas; and his work has done much to reveal the relationship between the two points of view and profoundly influenced the trend of subsequent research.

Although the two theorems of Weierstrass are now classical we give proofs of them and of some of the well-known results due to Cauchy on which the development of the subject depends. We shall also have to make use of certain inequalities obtained in the course of the proof of Weierstrass' first theorem.

1. The Laurent expansion about an isolated singularity. — Let $F(z)$ be a uniform function and let the point $z=a$ be a singular point of the function $F(z)$. We will suppose the point $z=a$ to be an isolated singularity; that is to say that we can find a circle C of finite radius α and centre a such that $F(z)$ has no other singularity in the circle or on its circumference. If now C′ is another circle of centre a and radius $\beta<\alpha$, $F(z)$ will be regular on the circles C and C′ and in the annular region between them, and we may apply Cauchy's theorem to the function in this region. For an interior point x of the annular region we thus obtain the equation

$$F(x)=\frac{1}{2\pi i}\int_{C}\frac{F(z)}{z-x}\,dz-\frac{1}{2\pi i}\int_{C'}\frac{F(z)}{z-x}\,dz,$$

both integrals being taken in the positive sense. From this it follows that $F(x)$ *is the sum of two convergent series, one in positive powers of* $(x-a)$ *and the other in negative powers of* $(x-a)$:

$$F(x)=f\left(\frac{1}{x-a}\right)+\Phi(x-a) \tag{1, 1}$$

where

$$f(u)=\sum_{1}^{\infty}c_n u^n,\qquad \Phi(u)=\sum_{0}^{\infty}c_{-n}u^n.$$

The coefficients c_i are given by Cauchy's equations

$$\begin{cases} c_n=\dfrac{1}{2\pi i}\displaystyle\int_{C'}F(z)(z-a)^{n-1}\,dz,\\[2ex] c_{-n}=\dfrac{1}{2\pi i}\displaystyle\int_{C}\dfrac{F(z)}{(z-a)^{n+1}}\,dz.\end{cases} \tag{1, 2}$$

The values of these coefficients are unaffected by any change in the radii of the circles C and C′, provided that C contains no singularity other than a. Further, since the power series $\Phi(x-a)$ is convergent for $\beta < |x-a| < \alpha$, it is convergent for $|x-a| < \alpha$. Also $f(u)$ is an integral function. For the power series $f(u)$ converges for $|u| > \frac{1}{\beta}$ for all values of β, and so for all values of u.

Equation (1, 1), *the right-hand side of which is known as the Laurent expansion of* $F(z)$ *about the point* a, *is therefore valid inside any circle* C *of centre* a *and containing no other singular point of the function.*

In the case where the function $f(u)$ reduces to a polynomial the point $z = a$ is a pole of $F(z)$ and the properties of the function in its neighbourhood are well known. We shall consider the case in which $f(u)$ does not reduce to a polynomial. The singular point is then an *isolated essential singularity.*

In most cases we shall suppose the essential singularity to be at infinity, when we shall have

$$F(z) = f(z) + \Phi(z) \tag{1, 3}$$

where $f(z)$ *is an integral function and* $\Phi(z)$ *a function which vanishes and is regular at infinity.* In future we shall write

$$f(z) = \sum_{0}^{\infty} c_n z^n = c_0 + c_1 z + \ldots,$$

$$\Phi(z) = \sum_{-1}^{-\infty} c_n z^n = \frac{c_{-1}}{z} + \ldots,$$

the first expansion being valid for all z and the second for $|z| > R_0$, where R_0 is some fixed number. For all values of n we have

$$c_n = \frac{1}{2i\pi} \int_\Gamma F(z) z^{-n-1} dz,$$

Γ being some circle of centre $z = 0$ and radius greater than R_0. The modulus of c_n will be denoted by C_n.

2. Theorems of Cauchy and Liouville. — Now let us consider a function $\Psi(z)$ which is regular in a circle C, $|z| \leqslant r$, and let

$$\Psi(z) = \sum_0^\infty a_n z^n.$$

The coefficients are given by Cauchy's equations (1, 2); so that

$$2\pi i a_n = \int_C \Psi(z) z^{-n-1} dz.$$

We may express these quantities in terms of their moduli and arguments by writing

$$z = re^{i\varphi}, \qquad a_n = A_n e^{i\theta_n},$$
$$\Psi(z) = G(r, \varphi) e^{i\Theta(r, \varphi)},$$

which gives us

$$2\pi A_n r^n e^{i\theta_n} = \int_0^{2\pi} G(r, \varphi) e^{i[\Theta(r, \varphi) - n\varphi]} d\varphi.$$

Hence

$$2\pi A_n r^n = \int_0^{2\pi} G(r, \varphi) \cos [\Theta(r, \varphi) - n\varphi - \theta_n] d\varphi$$

Now the functions $G(r, \varphi)$ and $\Theta(r, \varphi)$ are continuous fonctions of φ. $G(r, \varphi)$ has therefore a maximum on C, which it effectively attains; and so, if $M(r)$ is this maximum, the inequality

$$(1, 4) \qquad A_n r^n \leqslant M(r)$$

is true for all values of n. It follows from the property of continuity that, for a given value of n, this inequality can only be replaced by an equality if

$$G(r, \varphi) = M(r), \qquad \Theta(r, \varphi) = n\varphi + \theta_n + 2k\pi,$$

for all values of φ. We should then have $A_n = 0$ for all other values of n. In particular

$$A_0 = \Psi(0) < M(r) \tag{1, 5}$$

unless $\Psi(z)$ is a constant.

We are now in a position to prove Cauchy's theorems.

THEOREM 1. — *The maximum of the modulus of a function $g(z)$, which is regular in a closed connected region* D, *bounded by one or more curves* C, *is attained on the boundary* (1).

Since the modulus is a continuous function it certainly attains its maximum in at least one point P of the closed region D. If the point P were an interior point of D we could find a circle, of centre P, lying entirely in D, and the value of $|g(z)|$ at the centre of this circle would be at the least equal to its maximum on the circumference. But, unless $g(z)$ is a constant, this contradicts the inequality (1, 5).

As an immediate corollary of this proposition, in conjunction with the remark made in connection with (1, 4), we have this further result :

THEOREM 2. — *If $f(z)$ is a complete* (2) *integral function, the maximum $M(r)$ of the modulus of $f(z)$ for $|z| = r$, is an increasing function of r and*

$$C_n r^n < M(r) \tag{1, 6}$$

for all values of r and n.

We may observe that it follows from this proposition that $\sqrt[n]{C_n}$ tends to zero as n tends to infinity.

---

(1) A "region" is a domain with possibly some or all of its frontier points, a "domain" being defined as a set of points all of which are interior points of the set. In the case of a single variable we shall regard a "segment" as closed and an "interval" as open.

(2) A function in which there is a sequence of the coefficients c_n different from zero.

In the case of a complete integral function there is a sequence of coefficients C_n which are not zero, and so we may deduce from the inequality (1, 6) the following corollary, which is known, in a slightly different form, as *Liouville's theorem* :

COROLLARY 2. — *If $f(z)$ is an integral function and p any positive number whatever, the ratio*

$$\frac{M(r)}{r^p}$$

ultimately surpasses any assigned number.

Let us now consider the case of a function $F(z)$ for which the point at infinity is an isolated essential singularity. $F(z)$ is the sum of an integral function and a function which tends to zero as $r = |z|$ tends to infinity. If by $M_1(r)$ we denote the maximum modulus of $F(z)$ for $|z| = r$, and by $M(r)$ that of $f(z)$ [equation (1, 3)] and if γ is any positive number, we shall have, for all sufficiently large values of r,

$$M_1(r) > M(r) - \gamma,$$

so that the ratio

$$\frac{M_1(r)}{r^p}$$

also ultimately surpasses any assigned number.

On the other hand we may apply theorem 1 to the function $F(z)$ and the region D defined by the inequalities $R_0 < R \leqslant |z| \leqslant R'$. The maximum of the modulus is attained on one of the bounding circles, and therefore. for all sufficiently large values of R' on the outer circle. So $M_1(r)$ is ultimately an increasing function of r. Since when k is an integer $F(z)z^{-k}$ is of the same form as $F(z)$, we have the following parallel result.

THEOREM 3. — *If $F(z)$ has an isolated essential singularity at infinity, the ratio*

$$\frac{M_1(r)}{r^p}$$

increases indefinitely after a certain value of r, for any fixed value of p.

Finally we observe that the inequality (1, 6) is satisfied by a function $F(z)$ if we suppose n to be positive and replace $M(r)$ by $M_1(r)$.

The property established in theorem 3, which is one of the initial properties of $M_1(r)$, brings into prominence the sharp distinction which exists between the case of a pole and that of an essential singularity.

3. The indetermination of a uniform fónction in the neighbourhood of an isolated essential singularity. — If we consider a sequence of points whose sole limiting point is a pole of a uniform function, the sequence of the moduli of the function at these points always tends to infinity as its limit. This is no longer true as we approach an essential singularity. Let us assume that this essential singularity is at infinity. Then we can still find a sequence of points in which the modulus tends to infinity. In fact it is sufficient to take those points at which $|F(z)| = M_1(|z|)$ on a sequence of circles of indefinitely increasing radius. Now consider a finite number a. It may happen that $F(z)$ assumes the value a an infinity of times. In this case, since the zeros of a regular function are isolated, the points at which $F(z) = a$ have their sole limiting point at infinity. But if $F(z)$ only assumes the value a a finite number of times the function

$$\Phi(z) = \frac{1}{F(z) - a}$$

will be regular for $|z| > R_1$, R_1 being the modulus of the most distant zero of $F(z) - a$. The point at infinity must be an essential singularity of $\Phi(z)$, for if it were a regular point or a pole it would be a pole or regular point of $F(z)$. There is therefore a sequence of points, whose sole limiting point is at infinity, in which $\Phi(z)$ tends to infinity and $F(z) - a$ tends to zero. We thus obtain Weierstrass' theorem.

THEOREM 4. — *In the case of a uniform function we can always find a sequence of points, whose limiting point is an essential singularity, in which the function tends to any assigned limit.*

This theorem of Weierstrass was completed by Picard, who showed that a uniform function actually assumes every value, with only one

possible exception, an infinity of times in the neighbourhood of an isolated essential singularity. Much of our work will be devoted to the study of Picard's theorem and its generalisations.

4. Convergent sequences of regular functions. — There is another fundamental theorem due to Weierstrass.

THEOREM 5 (¹). — *Let*

$$\varphi_1(z), \quad \varphi_2(z), \quad \ldots, \quad \varphi_n(z), \quad \ldots$$

be a sequence of functions which are regular in a closed connected region D *bounded by a contour* Γ. *If this sequence converges uniformly on the curve* Γ, *it converges uniformly throughout the closed region, and the limit of the sequence is a function* $\Phi(z)$ *which is regular in the open domain, and it derivative of order* p *is the limit of the sequence of* p*'th derivatives of* $\varphi_n(z)$.

Uniform convergence in the region D is practically immediate. For, by hypothesis, we have on the contour

$$|\varphi_{n+p}(z) - \varphi_n(z)| < \varepsilon$$

for all ε and $n > N(\varepsilon)$. By theorem 1 this inequality also holds at all interior points of D.

Now let $\Phi(z)$ be the limiting function of the sequence. $\Phi(z)$ is a continuous function. Let $\theta(z)$ be the regular function defined by the integral

$$\theta(x) = \frac{1}{2i\pi}\int_\Gamma \frac{\Phi(z)}{z-x}\,dz.$$

We can shew that $\theta(x) = \Phi(x)$. If x is an interior point of D Cauchy's theorem gives

$$\varphi_n(x) = \frac{1}{2i\pi}\int_\Gamma \frac{\varphi_n(z)}{z-x}\,dz$$

(¹) Certain generalisations of this theorem are to be found in Montel's book, *Leçons sur les séries de polynômes d'une variable complexe.*

and, subtracting this from the last equation,

$$|\theta(x) - \varphi_n(x)| = \frac{1}{2\pi}\left|\int_\Gamma \frac{\Phi(z) - \varphi_n(z)}{z - x}\,dz\right| < \frac{L}{2\pi}\frac{\varepsilon}{d}$$

where L is the length of the contour and d the shortest distance from the point x to the contour. ε can be made as small as we please by taking n sufficiently great. We have therefore

$$\theta(x) = \lim \varphi_n(x) = \Phi(x),$$

and the first part of the theorem is proved.

The second part follows from another application of Cauchy's theorem. In fact

$$|\varphi_n^{(k)}(x) - \Phi^{(k)}(x)| = \frac{1}{2\pi}\left|\int_\Gamma \frac{\varphi_n(z) - \Phi(z)}{(z - x)^{k+1}}\,dz\right| < \frac{L}{2\pi}\frac{\varepsilon}{d^{k+1}}$$

which shows that $\varphi_n^{(k)}(z)$ tends uniformly to $\Phi^{(k)}(z)$ in the open region.

5. The construction of an integral function with assigned zeros. — We have already had occasion to make use of the fact that the zeros of a function F(z) which is regular, except at infinity, for $|z| > R_0$, are isolated points; every annular region $R_0 < R \leqslant |z| \leqslant R'$ contains a finite number of them. So if F(z) has an infinity of zeros they can be arranged as an infinite sequence in order of non-decreasing moduli. If there are several zeros having the same modulus these can be arranged in any order we please. Repeated zeros will occur with a frequency equal to their order of multiplicity. We denote by a_n the n'th zero of this sequence and its modulus by r_n. The sequence

$$r_1, \quad r_2, \quad \ldots, \quad r_n, \quad \ldots$$

is non-decreasing and its sole limiting point is $+\infty$. In the particular case of an integral function which vanishes at the origin we shall exclude the number o from this sequence.

It as been shown by Weierstrass that if we are given any sequence of numbers a_n whose sole limiting point is at infinity, then there is

an integral function with zeros at these points and at these points only. In order to do this Weierstrass constructed an infinite product of regular functions each one of which vanishes for a single value a_n and for this value only. To ensure the convergence of the product he employed as *primary factor*, vanishing for $u = 1$, the expression

$$E(u, p) = (1-u)e^{u+\frac{u^2}{2}+\cdots+\frac{u^p}{p}}$$

where the exponent of e consists of the first p terms of the expansion of $-\log(1-u)$; so that, for $|u| < 1$,

$$E(u, p) = e^{-\frac{u^{p+1}}{p+1}-\frac{u^{p+2}}{p+2}-\cdots}$$

If we take $|u| \leqslant \frac{1}{k} < 1$, the exponent of e in this last equation will be less in modulus than

$$\frac{|u|^{p+1}}{p+1}\left(1+\frac{1}{k}+\cdots\right) = \frac{|u|^{p+1}}{p+1}\,\frac{k}{k-1}.$$

We may therefore state the following result:

Lemma 6. — If $|u| \leqslant \frac{1}{k} < 1$, *then*

$$(1, 7) \qquad E(u, p) = e^{\theta\frac{k}{k-1}\frac{|u|^{p+1}}{p+1}},$$

where $|\theta| < 1$.

There is one other property of the sequence whose importance will appear directly. Suppose that, as above, the points a_n have the point at infinity as their sole limiting point and that the sequence of numbers r_n is arranged in the manner we described. r_1 is supposed not to be zero. *Then with the sequence r_n we can associate a sequence of positive integers p_n such that the series*

$$(1, 8) \qquad \sum_1^\infty \left(\frac{r}{r_n}\right)^{p_n}$$

converges for all values of r.

In fact it is sufficient to take for p_n the integral part of

$$(1+\alpha)\frac{\log n}{\log r_n}+1 \qquad (\alpha>0).$$

We then have, for $r_n > r$,

$$\left(\frac{r}{r_n}\right)^{p_n} \leqslant \left(\frac{r}{r_n}\right)^{\frac{\log n}{\log r_n}(1+\alpha)} = \frac{1}{n^{(1+\alpha)v_n}} \qquad \left(v_n = 1 - \frac{\log r}{\log r_n}\right).$$

v_n tends to 1 as n tends to infinity. So, for sufficiently large values of n, $(1+\alpha)v_n$ will be greater than 1 and the series (1, 8) will converge.

To construct an integral function with zeros at the points a_n we consider the product

$$P_m(z) = \prod_1^m E\left(\frac{z}{a_n}, p_n - 1\right)$$

where p_n is chosen so that the series (1, 8) is convergent for all values of r. This product is plainly an integral function vanishing at the points a_n of index less than or equal to m and at these points only. We shall show that the sequence of functions $P_m(z)$ converges uniformly in every finite region. To do this we establish uniform convergence in a circle $|z| \leqslant R$ of arbitrary radius. Let k be a fixed number greater than 1 and let N be defined by the inequalities

$$r_N \leqslant kR < r_{N+1}.$$

For m greater than N we may write

$$P_m(z) = P_N(z)\,Q_m(z)$$

where Q_m is the product of factors whose rank lies between N and $m+1$. In virtue of Lemma 6 and the convergence of the series (1, 8) for the value R,

$$(1, 9) \quad \log|Q_m(z)| < \frac{k}{k-1}\sum_{N+1}^{m}\left(\frac{r}{r_n}\right)^{p_n} < \frac{k}{k-1}\sum_{N+1}^{\infty}\left(\frac{R}{r_n}\right)^{p_n} = \log K$$

$$(|z| = r \leqslant R),$$

where K is a constant depending only on R. Therefore, if M is the maximum modulus of $P_N(z)$ in the circle $|z| \leqslant R$, we shall have in this circle, and for all values of m,

$$|P_m(z)| < KM.$$

Hence, if $m' > m > N$,

$$|P_{m'}(z) - P_m(z)| < KM \left| \frac{Q_{m'}(z)}{Q_m(z)} - 1 \right|.$$

A further appeal to Lemma 6 and the convergence of the series (1, 8) gives

$$\frac{Q_{m'}(z)}{Q_m(z)} = e^{\theta w_{m'}} \qquad (|\theta| < 1),$$

where

$$w_{m'} = \frac{k}{k-1} \sum_{m+1}^{m'} \left(\frac{r}{r_n}\right)^{p_n} < \frac{k}{k-1} \sum_{m+1}^{\infty} \left(\frac{R}{r_n}\right)^{p_n} = \frac{k}{k-1} \varepsilon_m,$$

ε_m denoting the remainder after the first m terms of the series (1, 8). Thus

$$\left| \frac{Q_{m'}(z)}{Q_m(z)} - 1 \right| = |e^{\theta w_{m'}} - 1| < e^{w_{m'}} - 1 < e^{\frac{k}{k-1}\varepsilon_m} - 1$$

which may be made arbitrarily small provided m is taken sufficiently large, wherever z may be in the circle $|z| \leqslant R$. The sequence of functions $P_m(z)$ is therefore uniformly convergent in this circle and its limit, which we shall denote by $P(z)$, is a function regular in the circle.

Moreover this function $P(z)$ is not a constant, for in the circle of radius R we have

$$\log |Q_m(z)| > -\frac{k}{k-1} \sum_{N+1}^{\infty} \left(\frac{R}{r_n}\right)^{p_n} = \log \frac{1}{K}$$

and so

$$(1, 10) \qquad |P(z)| \geqslant |P_N(z)| \frac{1}{K}.$$

Therefore $P(z)$ does not vanish in the circle under consideration unless $P_N(z)$ vanishes; that is to say at the points a_n and at these points only.

We may change the order of the factors in the product $P(z)$ in any way we please, for the investigation of $Q_{m'}(z)/Q_m(z)$ carried out above shews that the product of any number of primary factors of rank higher than a number $N(R, \varepsilon)$ differs from 1 by less than ε, for $|z| \leqslant R$. Further any product of factors is uniformly bounded. The difference of two products which both include the first $N(R, \varepsilon)$ factors will therefore be less than K_ε. The property of *absolute convergence* is thus established and the proof of Weierstrass' theorem complete([1]).

THEOREM 6. — *If the sequence of integers p_n is such that the series* (1, 8) *is convergent for all values of r, the infinite product*

$$P(z) = \prod_1^\infty E\left(\frac{z}{a_n}, p_n - 1\right)$$

is absolutely and uniformly convergent in every finite region, and defines an integral function with zeros at the points a_n and at these points only.

We may observe that if there is an integer p such that the series

$$\sum_1^\infty \frac{1}{r_n^{p+1}}$$

converges, we may take $p_n = p + 1$, when we have simply

$$P(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \dots + \frac{z^p}{p a_n^p}}.$$

([1]) The proof is much simplified if the properties of infinite products are assumed.

6. Factorisation of a function F(z). — Let us suppose that our function $F(z)$ regular for $|z| \geqslant R_0$, except for an isolated essential singularity at infinity, has an infinity of zeros. We can construct a Weierstrassian product $P(z)$ with the same zeros to the same order of multiplicity. The function

$$G_1(z) = \frac{F'(z)}{F(z)} - \frac{P'(z)}{P(z)}$$

will also be regular for $|z| \geqslant R_0$, because a zero of $F(z)$, of order q, gives rise to a simple pole of residue q both for $F'(z)/F(z)$ and $P'(z)/P(z)$. Integrating $G_1(z)$ from z_0 to z we obtain

$$\int_{z_0}^{z} G_1(z)dz = G_2(z) + \alpha \log \frac{z}{z_0},$$

when $G_2(z)$ is again a function which can be expanded in a Laurent series for $|z| \geqslant R_0$, and α is the coefficient of $\frac{1}{z}$ in $G_1(z)$. Thus it follows that

$$\frac{F(z)}{P(z)} = \frac{F_0}{P_0}\left(\frac{z}{z_0}\right)^{\alpha} e^{G_2(z)} = z^{\alpha} e^{G(z)}$$

where $G(z)$ can be expressed in the form of a Laurent series for $|z| \geqslant R_0$. As the functions F, P and G are uniform, α is either a positive or negative integer or zero. Moreover

$$G(z) = g(z) + \Phi_1\left(\frac{1}{z}\right),$$

$g(z)$ being an integral function and $\Phi_1\left(\frac{1}{z}\right)$ regular for $|z| \geqslant R_0$, vanishing and regular at infinity. The function $\Phi\left(\frac{1}{z}\right) = e^{\Phi_1\left(\frac{1}{z}\right)}$ is therefore also regular at infinity, where it assumes the value 1.

It is clear that our argument remains valid when the function $F(z)$ has only a finite number of zeros. $P(z)$ will then be a polynomial. We have thus established the following corollary :

Corollary 6. — *Suppose that the only singularity of* F(z) *in the domain* $|z| \geqslant R_0$ *is an essential singularity at infinity. Then the function* F(z) *can be expressed in the form*

$$(1, 11) \qquad F(z) = z^{\alpha} P(z) \Phi\left(\frac{1}{z}\right) e^{g(z)},$$

where P(z) *is a polynomial or Weierstrassian product corresponding to the zeros of* F(z); α *is a positive or negative integer or zero;* $g(z)$ *is an integral function; and* $\Phi\left(\frac{1}{z}\right)$ *is a function which is regular and does not vanish throughout the domain* $|z| \geqslant R_0$ *and is equal to* 1 *at infinity.*

In the particular case where F(z) reduces to an integral function $f(z)$, $\Phi\left(\frac{1}{z}\right)$ reduces to 1 and α is positive or zero. In fact

$$(1, 12) \qquad f(z) = z^{\alpha} P(z) e^{g(z)}.$$

This factorisation depending on the zeros is not completely determinate, for we have seen that any sequence of numbers p_n may be replaced by a sequence of larger numbers. Later on we shall shew that in certain cases it is possible to obtain a decomposition into factors which is strictly determinate.

CHAPTER II

The function M(r) and the coefficients in the expansion f $f(z)$.

We have indicated (I. 2) certain properties of the maximum modulus of an integral function $f(z)$ and we have established a relation between this function and the coefficients in the Taylor series of $f(z)$ (I, 6). We proceed to complete these results. It will be shewn that the maximum modulus M(r) is a continuous function and *differentiable in adjacent intervals* (¹) and that log M(r) is a convex function of log r. Our principal object however in the present chapter is to investigate the relations existing between the moduli $C_n r^n$ of the terms in the Taylor expansion and the function M(r). We shall, in fact, shew that the logarithm of the greatest of the terms $C_n r^n$ is asymptotically equivalent to log M(r).

This leads us to the distinguish a certain class of functions, those of finite order, for which the relation appears in a peculiarly simple form. Narrowing down our classification still further we shall recognise in this class functions which are of *regular growth*. It is with such functions as these that we are generally concerned in applications.

I. — The growth of the functions M(r) and A(r).

Consider an integral function $f(z)$, or simply a function $\Phi(z)$ regular in a circle of radius R with its centre at the origin. The function $e^{\Phi(z)}$ is also regular in the same circle and its maximum modu-

(¹) A function is said to be differentiable in adjacent intervals in a certain range (or from o to ∞) when the range can be split up into non-evanescent intervals in each of which the function is differentiable whilst it has right and left hand derivatives at the left and right-hand end points.

lus for $|z| = r$ is equal to $e^{A(r)}$, where A(r) denotes the maximum of the real part of $\Phi(z)$ for $|z| = r$. It therefore follows from theorem (1) that this function A(r) is also an increasing function. Similarly, if we denote by $-B(r)$ the minimum real part of $\Phi(z)$, it follows from a consideration of the function $e^{-\Phi(z)}$ on the circle $|z| = r$ that B(r) is an increasing function. There are analogous results for the maximum and minimum values of the imaginary part of $\Phi(z)$. We have seen that when a function is expressed as a Weierstrassian product a factor $e^{g(z)}$ is introduced. A(r) and B(r) play an important part in the study of these functions $e^{g(z)}$.

1. Comparison of the functions M(r) and A(r). — Hadamard, using only the simplest properties of trigonometrical series, shewed that Cauchy's inequality (1, 6) and the deductions from it can be extended to the function A(r).

Let

$$\Phi(z) = \sum_{0}^{\infty} c_n z^n$$

be a function regular in the circle $|z| \leqslant r$. Writing

$$z = re^{i\varphi}, \qquad c_n = c_n' + ic_n'', \qquad \Phi(z) = P(r, \varphi) + iQ(r, \varphi)$$

we have

$$P(r, \varphi) = c_0' + \sum_{1}^{\infty} [c_n' \cos(n\varphi) - c_n'' \sin(n\varphi)] r^n.$$

This series is uniformly convergent for all values of φ since, by hypothesis, the series $\Sigma c_n r^n$ is convergent. We may therefore multiply through by $\cos(n\varphi)$ or $\sin(n\varphi)$ and integrate term by term between the limits 0 and 2π. We thus find

$$\pi r^n c_n' = \int_0^{2\pi} P(r, \varphi) \cos(n\varphi) d\varphi,$$

$$\pi r^n c_n'' = -\int_0^{2\pi} P(r, \varphi) \sin(n\varphi) d\varphi.$$

Multiplying the second of these equations by i and adding we obtain

$$\pi r^n c_n = \int_0^{2\pi} P(r, \varphi) e^{-in\varphi} d\varphi .$$

Hence

$$\pi r^n C_n \leqslant \int_0^{2\pi} |P(r, \varphi)| \, d\varphi$$

We have also

$$2\pi c_0' = \int_0^{2\pi} P(r, \varphi) d\varphi ,$$

and so, by addition,

$$\pi(C_n r^n + 2c_0') \leqslant \int_0^{2\pi} (|P(r, \varphi)| + P(r, \varphi)) d\varphi .$$

Now the integrand on the right is zero when $P(r, \varphi)$ is negative or zero, and equal to $2P(r, \varphi)$ when $P(r, \varphi)$ is positive. The right hand side of this inequality is therefore zero when $A(r)$ is negative or zero and less than $4\pi A(r)$ when this number is positive. Hence we have

Theorem 7. — *If the function $\Phi(z)$ is regular in the circle $|z| \leqslant r$, and if $A(r)$ is its maximum real part on the circumference of this circle, then, for all positive values of n, the numbers $C_n r^n$ are less than or equal to the greatest of the two numbers $-2c_0'$ and $4A(r) - 2c_0'$.*

In the case of an integral function, if c_p is the first coefficient which is not zero ($p > 0$), $C_p r^p + 2c_0'$ will certainly be positive for all sufficiently large values of r. The same is therefore true of $A(r)$ and we have the following corollary :

Corollary 7(a). — *In the case of an integral function,*

(2, 1) $$C_n r^n \leqslant 4A(r) - 2c_0'$$

for all positive values of n and for all $r > r_0$.

The same results are evidently true for B(r), except that c_0 will be changed into $-c_0'$.

From this inequality (2, 1) we can deduce a theorem analogous to Liouville's theorem.

COROLLARY 7(b). — *If* D *and* q *are two fixed positive numbers and if the real part of an integral function* $f(z)$ *is algebraically less than* Dr^q *on a sequence of circles* $|z| =$ constant *of indefinitely increasing radius, then* $f(z)$ *is a polynomial of degree not greater than* q.

For, if r is the radius of one of these circles and if this radius is sufficiently large, we have

$$C_n \leqslant 4Dr^{q-n} - 2c_0'r^{-n}.$$

But if $n > q$ the right-hand side tends to zero as r tends to infinity; and so C_n is zero for $n > q$.

Borel has shewn that the inequality (2, 1) leads very simply to a relation between M(r) and A(r). We can write this inequality in the form

$$C_n r^n \leqslant [4A(R) - 2c_0']\left(\frac{r}{R}\right)^n$$

and, taking $r < R$, it follows that

$$\begin{aligned} M(r) &\leqslant \sum C_n r^n \leqslant C_0 + \frac{r}{R-r}[4A(R) - 2c_0'] \\ &< \frac{R}{R-r}[4A(R) + C_0 - 2c_0'] \leqslant \frac{R}{R-r}[4A(R) + 3C_0]. \end{aligned}$$

Hence we have Borel's theorem.

THEOREM 8. — *If* $f(z)$ *is an integral function, then*

$$(2,\ 2) \qquad B(r) \leqslant M(r) < \frac{R}{R-r}[4A(R) + 3C_0] \qquad (R > r)$$

for all $r > r_0$.

A slightly improved form of this inequality has been found by Caratheodory, who also proved that it is in fact true for all values of r. Without going into details we may point out here that the hypothesis that $f(z)$ is an integral function enables us to prove that ultimately $A(r)$ is positive and the inequality (2, 1) valid. Otherwise it does not affect the argument.

Observing that if a regular function vanishes at the origin, then $A(r)$ and $B(r)$, which are increasing functions, are positive, we have as a corollary :

Corollary 8. — *If the function* $\Phi(z)$ *is regular for* $|z| \leqslant R$ *and zero at the origin, then*

$$(2, 3) \qquad B(r) \leqslant M(r) \leqslant \frac{4r}{R-r} A(R) \qquad (r < R).$$

Clearly we can interchange $A(r)$ and $B(r)$ in the foregoing arguments and propositions. It is easy to find analogous results for functions $F(z) = f(z) + \Phi\left(\frac{1}{z}\right)$ with an isolated essential singularity at infinity.

2. Hadamard's theorem on the growth of $\log M_1(r)$. — We have seen that for sufficiently large values of r the ratio $M_1(r)/r^p$ is an indefinitely increasing function. A more precise result has been obtained by Hadamard.

Theorem 9. — *The function* $\log M_1(r)$ *is a continuous, convex and ultimately increasing function of* $\log r$.

Let us write

$$X = \log r, \qquad V(X) = \log M_1(r),$$

and consider three values of X,

$$X_1 < X_2 < X_3.$$

The number h being defined by the equation

$$V(X_3) - hX_3 = V(X_1) - hX_1$$

the function $F(z)\,z^{-h}$ is in general multiform in the annular region $e^{X_1} \leqslant |z| \leqslant e^{X_3}$, but its different determinations are regular at all points of this region and its modulus is uniform. This modulus has a maximum value which, by the proof of theorem 1, is attained on one of the two bounding circles of the annulus, and we have therefore

$$V(X_2) - hX_2 < V(X_1) - hX_1 = V(X_3) - hX_3;$$

whence, substituting for h,

$$\frac{V(X_2) - V(X_1)}{X_2 - X_1} < \frac{V(X_3) - V(X_1)}{X_3 - X_1} < \frac{V(X_3) - V(X_2)}{X_3 - X_2}.$$

The convexity of the function is expressed by these inequalities. If we plot the points of coordinates X, V(X), the point of index 2 lies below the line joining the points of indices 1 and 3. It can be seen further that V(X) has a right-hand derivative. For, if k is positive, $[V(X + k) - V(X)]/k$ is a decreasing function of k, bounded below, and consequently has a limit for $k = 0$. Similarly there is a left-hand derivative $V'(X - 0)$ which does not exceed $V'(X + 0)$. Finally letting $X_1 \to X_3$ we observe that there is in general a derivative $V'(X)$ which is an increasing function. Since we already know that $M_1(r)$ is ultimately an increasing function of r, the theorem is proved([1]). Either V(X) is always an increasing function or else it starts as a decreasing function and then increases. Integral functions are of the first class. So are functions such as

$$F(z) = e^z\left(A + \frac{1}{z} + \ldots + \frac{1}{n^2 z^n} + \ldots\right)$$

provided that A is sufficiently large. The function e^z/z (however) is of the second class.

([1]) Hardy has proved that if $\Phi(z)$ is regular for $0 < |z| = r < R$, and if

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\Phi(re^{i\theta})|\,d\theta,$$

then log I(r) is also a convex increasing function of log r ($0 < r < R$) See G. H. Hardy. *Proc. London Math. Soc.*, 1915.

3. Results due to Blumenthal: — That the function $M_1(r)$ is differentiable in adjacent intervals was shewn by Blumenthal. Using the same notation as in the proof of theorem 7 we write

$$F(z) = P(r, \varphi) + iQ(r, \varphi),$$

where

$$P(r, \varphi) = \sum_{-\infty}^{+\infty} (c_n' \cos n\varphi - c_n'' \sin n\varphi) r^n,$$

$$Q(r, \varphi) = \sum_{-\infty}^{+\infty} (c_n' \sin n\varphi + c_n'' \cos n\varphi) r^n.$$

The square of the modulus of $F(z)$ is then given by the equation

$$m(r, \varphi) = \{P(r, \varphi)]^2 + [Q(r, \varphi)]^2.$$

Now the series of positive terms

$$\sum_{0}^{\infty} C_n (e^{|\varphi|} r)^n + \sum_{1}^{\infty} C_{-n} (e^{-|\varphi|} r)^{-n},$$

which is a dominant for both the series P and Q, is convergent for $r \geqslant R_0 e^{2\pi}$ and $0 \leqslant |\varphi| \leqslant 2\pi$. The series P and Q may therefore be arranged in order of ascending powers of φ and they represent functions of r and φ, regarded as complex variables, analytic and regular in the domain $r \geqslant R_0 e^{2\pi}$, $|\varphi| \leqslant 2\pi$. The same is therefore true of the function $m(r, \varphi)$ and of $\dfrac{\partial m(r, \varphi)}{\partial \varphi}$, its derivative with respect to φ.

We proceed to a study of the set of real values r, φ for which $\dfrac{\partial m(r, \varphi)}{\partial \varphi}$ vanishes, and then, with the aid of Weierstrass' theorem on implicit functions, of the corresponding set of values of $m(r, \varphi)$. The investigation depends on certain preliminary lemmas.

Lemma 10. — *Given an annular region* $R \leqslant |z| \leqslant R'$, *there is a function* $U(z)$ *with the following properties :*

(i) $U(z)$ *is regular throughout the region and vanishes at a given interior point.*

(ii) $U(z)$ *has a pole and does not vanish in the neighbouring annulus* $R' \leqslant |z| \leqslant R'^2/R$.

(iii) *The modulus of* U *is constant on each of the circles* $|z| = R$ *and* $|z| = R'$.

$U(z)$ is known as a Green's function for the region.

We may suppose that $R = 1$ and $R' = k > 1$. Let α be the given point at which $U(z)$ vanishes and let α' be the inverse of α with respect to the unit circle (α, α' lie on the same radius and $|\alpha\alpha'| = 1$). By theorem 6 and in virtue of the convergence of the series Σk^{-2n}, the infinite product

$$P(z) = \prod_{0}^{\infty} \left(1 - \frac{z}{k^{2n}}\right),$$

defines an integral function with zeros at the points $z = 1, \ldots, k^{2n}, \ldots$. The function

$$S(z) = P(z)\, P\left(\frac{1}{k^2 z}\right)$$

is regular in any annulus described about the origin as centre and its zeros are the points k^{2n} $(n = 0, \pm 1, \ldots)$. Now consider the function

$$U(z) = S(z/\alpha) / S(z/\alpha').$$

The points $z = \alpha k^{2n}$ are zeros and the points $z = \alpha' k^{2n}$ poles of $U(z)$ and it has no other singularities in the finite part of the plane. Further $|U(z)|$ is constant on each of the two circles $|z| = 1$ and $|z| = k$. For the function may be expressed in either of the two forms

$$U(z) = -\frac{\alpha'}{z} \prod_{0}^{\infty} \left(\frac{1 - \dfrac{z}{\alpha k^{2n}}}{1 - \dfrac{\alpha'}{z k^{2n}}} \right) \prod_{1}^{\infty} \left(\frac{1 - \dfrac{\alpha}{z k^{2n}}}{1 - \dfrac{z}{\alpha' k^{2n}}} \right)$$

$$= \frac{k^2 \alpha'^2}{z^2} \prod_{0}^{\infty} \left(\frac{1 - \dfrac{z}{\alpha k^{2n}}}{1 - \dfrac{\alpha' k^2}{z k^{2n}}} \right) \prod^{\infty} \left(\frac{1 - \dfrac{\alpha}{z k^{2n}}}{1 - \dfrac{z}{\alpha' k^{2n+2}}} \right)$$

and in the first of these products every factor is constant in modulus on the circle $|z| = 1$ and in the second the same is true on the circle $|z| = k$. Finally $U(z)$ is regular and vanishes at the point α in the annulus $1 \leqslant |z| \leqslant k$, and has a pole ($z = k^2\alpha'$), but no zero, in the neighbourhing annulus $k \leqslant |z| \leqslant k^2$. The conditions of the lemma are therefore fulfilled.

LEMMA 10'. — *The real points of the circle $|z| = r$ at which $\frac{\partial m}{\partial \varphi}$ vanishes are finite in number, for all values of r with one possible exception.*

For a given value of r ($|r| > R_0 e^{2\pi}$) the function $\frac{\partial m(r, \varphi)}{\partial \varphi}$ is regular in the circle $|\varphi| \leqslant 2\pi$. So, unless it is identically zero, it has a finite number of zeros, and in particular of real zeros, in this circle. If $\frac{\partial m(r, \varphi)}{\partial \varphi}$ is identically zero the function $m(r, \varphi)$ will be constant on the circle. To prove the lemma we shall shew that $m(r, \varphi)$ cannot be constant on two circles $|z| = R$, $|z| = R'$.

Suppose that $|F(z)|$ is constant on the two circles $|z| = R$ and $|z| = R'$, and let $\alpha_1, \alpha_2, \ldots, \alpha_p$, be the zeros of $F(z)$ between these circles. We can construct functions $U(z, \alpha_1)$, $U(z, \alpha_2)$, ..., $U(z, \alpha_p)$, which vanish at these points and satisfy the conditions of lemma 10. The function

$$V(z) = \frac{F(z)}{U(z, \alpha_1) \ldots U(z, \alpha_p)}$$

will be regular and different from zero in the annulus $R \leqslant |z| \leqslant R'$ and its modulus will be constant on the two bounding circles. Further h can be determined so that the modulus of $V(z)z^h$ will be the same on these two circles. So, if M denotes this modulus, we have

$$|V(z)z^h| \leqslant M$$

at all points of the annulus. Similarly, since $\frac{1}{V(z)}$ in also regular in this region,

$$\left|\frac{1}{V(z)} z^{-h}\right| \leqslant \frac{1}{M}.$$

Consequently

$$|V(z)z^h| = M$$

and it follows that $V(z)z^h$ is constant. But this would imply that

$$F(z) = Kz^{-h}U(z, \alpha_1) \ldots U(z, \alpha_p),$$

which is clearly impossible, since the functions U have poles in the annulus $R' \leqslant |z| \leqslant R''^2/R$. The lemma is therefore proved.

That the exceptional case can actually arise is seen by considering the function

$$P(z)/P\left(\frac{1}{z}\right).$$

In the particular case of integral functions a simpler argument, based on the Green's function for the circle instead of for the annulus, shews that the result is true for all values of r.

LEMMA 10''. — *No one of the set of real points (r, φ) at which $\dfrac{\partial m}{\partial \varphi} = 0$ and $m(r, \varphi) \neq 0$ can be isolated.*

Since the function $u = \log m(r, \varphi)$, being the real part of an analytic function, satisfies Laplace's equation, its derivative with respect to φ, $\dfrac{1}{m}\dfrac{\partial m}{\partial \varphi}$, also satisfies this equation.

Let P be a point (r_0, φ_0) at which $\dfrac{\partial m(r_0, \varphi_0)}{\partial \varphi} = 0$ and $m(r_0, \varphi_0) \neq 0$, and consider a circle C of centre P and radius c so small that $m(r, \varphi)$ is not zero in this circle or on its circumference. Then, by the formula of § (1) for the real part of a function at the centre of a circle,

$$\frac{1}{m(r_0, \varphi_0)} \frac{\partial m(r_0, \varphi_0)}{\partial \varphi} = 0 = \frac{1}{2i\pi} \int_{(C)} \frac{1}{m} \frac{\partial m}{\partial \varphi} ds$$

The function $\dfrac{\partial m(r, \varphi)}{\partial \varphi}$ must therefore vanish at least once on this circumference. Thus the lemma is proved.

Now consider those values of r which are greater than the possible exception of lemma 10'. For such values $\frac{\partial m(r, \varphi)}{\partial \varphi}$ has only isolated zeros on $r = \text{constant}$. Let (r_0, φ_0) be a point of the circle $|z| = r_0$ at which $m(r_0, \varphi_0)$ is a maximum. Then $\frac{\partial m(r_0, \varphi_0)}{\partial \varphi} = 0$ and, in the neighbourhood of this point,

$$\frac{\partial m(r, \varphi)}{\partial \varphi} = \sum \alpha_{k,l}(r - r_0)^k (\varphi - \varphi_0)^l .$$

By Weierstrass' theorem on implicit functions, the equation $\frac{\partial m(r, \varphi)}{\partial \varphi} = 0$, regarded as an equation in φ, has, in a domain $|r - r_0| < \delta$, $|\varphi - \varphi_0| < \delta$ a finite number of solutions, which are regular with respect to $(r - r_0)^{\frac{1}{\rho}}$, ρ being a certain integer. These solutions, which we write

$$\varphi = \varphi_0 + \chi\left((r - r_0)^{\frac{1}{\rho}}\right)$$

are the only solutions of the equation in this domain. In virtue of lemma 10" at least one of these functions is real. Let C_0 be one of the analytic arcs obtained from this solution and suppose that we substitute in $m(r, \varphi)$ the corresponding value of φ. We thus obtain

$$(\beta) \qquad m(r, \varphi) = m(r_0, \varphi_0) + \chi_1\left((r - r_0)^{\frac{1}{\rho}}\right)$$

where $\chi_1(u)$ is regular in the circle $|u| < \delta_1$ and zero at $u = 0$. Repeating this process for each of the arcs proceeding from every point of the circle $r = r_0$ at which $m(r_0, \varphi)$ has a maximum equal $[M(r_0)]^2$ we obtain a finite number of expressions similar to (β) and valid in $r_0 - \delta_2$, $r_0 + \delta_2$. And these give all the relevant values of the maxima of $m(r, \varphi)$ in a finite interval $r_0 - \delta_3$, $r_0 + \delta_3$. For, on those portions of the circle of radius r_0 exterior to the arcs $|\varphi - \varphi_0| < \delta$, $m(r_0, \varphi) < [M(r_0)]^2 - \gamma$, say. So, in virtue of the property of continuity, we can find a small annulus $r_0 - \delta_3$, $r_0 + \delta_3$, such that, in those portions exterior to the angles $|\varphi - \varphi_0| < \delta, m(r, \varphi) < [M(r_0)]^2 - \frac{1}{2}\gamma$

whereas those portions interior to these angles contain curves on which $m(r, \varphi) > [M(r_0)]^2 - \frac{1}{2}\gamma$.

The functions (β) are continuous and we now consider their differences taken in pairs. They constitute a finite set of regular functions of a single variable $(r - r_0)^{\frac{1}{s_0}}$ all zero at the origin. We can therefore find intervals $r_0 \leqslant r \leqslant r_0 + \delta_4$ and $r_0 - \delta_5 \leqslant r \leqslant r_0$ in which one of these functions (β) is greater than or equal all the others, though not necessarily the same function in each interval. In each of these intervals $M_1(r)$ is a branch of an analytic function of the variable $(r - r_0)^{\frac{1}{s_0}}$ and therefore has a derivative.

Thus to every sufficiently large value of r there corresponds a finite non-evanescent interval $r - \delta_5$, $r + \delta_4$, in which the maximum modulus of the function is attained on certain curves such as we have found. Further $M_1(r)$ is differentiable in this interval except possibly at r, where it has right and left-hand derivatives. It follows from the Heine-Borel theorem that any finite interval R, R′ can be covered by a finite number of intervals in which $M_1(r)$ has these properties. Thus we have

THEOREM 10. — *For all values of r greater than a certain number R the maximum modulus $M_1(r)$ is differentiable in adjacent intervals and is attained on certain arcs of curves, $\varphi = \chi_2\left((r - r_0)^{\frac{1}{s_0}}\right)$, where χ_2 is an analytic function, the number of these arcs in any annulus being finite.*

In the case of integral functions the theorem is true for all values of r.

Combining these properties of $M_1(r)$ we find that for $r > r_0 \geqslant R$

$$(2, 4) \qquad \log M_1(r) = \log M_1(r_0) + \int_{r_0}^{r} \frac{W(x)}{x} dx$$

where $W(x)$ is an indefinitely increasing function continuous in adjacent intervals. In the case of integral functions we may put $r_0 = 0$.

Blumenthal and subsequently Hardy have constructed examples shewing that the curve of the maximum modulus can actually shew discontinuities.

II. — The maximum term $m(r)$ and functions of finite order.

We shall confine ourselves in what follows to the case of integral functions, but the method can be extended without difficulty to cover the case of a function which is the sum of an integral function and a regular function $\Phi\left(\frac{1}{z}\right)$ vanishing at infinity.

Consider the sequence of the moduli of the terms in the expansion of $f(z)$,

$$C_0, \ C_1 r, \ \ldots, \ C_n r^n, \ \ldots.$$

This sequence tends to zero for all values of r. For every value of r there is therefore one term of this sequence which is greater than or equal to all the rest. This term (or one of these terms) we shall call the *maximum term for the given value* r, or simply the *maximum term*, and we shall denote its value by $m(r)$. It was proved by Borel that the functions $m(r)$ and $M(r)$ are of the same order of magnitude, in the sense that their logarithms are in general asymptotically equivalent. We shall shew that for a certain class of functions this is true without restriction. The general case will be dealt with in Chapter IV.

4. A general relation between $M(r)$ and $m(r)$. — As a systematic method of finding the maximum term we shall adopt the following procedure. Putting

$$\log C_n = -g_n$$

we have

$$(2, 5) \qquad \lim_{n=\infty} \frac{g_n}{n} = +\infty,$$

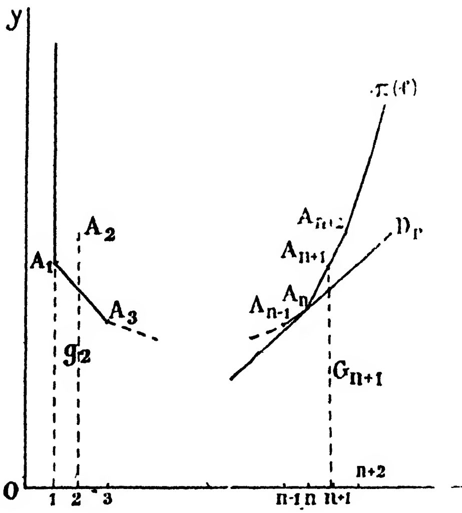
since $\sqrt[n]{C_n}$ tends to zero as n tends to infinity.

Taking axes of coordinates ox, oy let us plot the points A_n of coordinates n, g_n. When C_n is zero the y-coordinate of A_n is $+\infty$. It follows from (2, 5) that we can construct a Newton's polygon having

certain of the points A_n as its vertices whilst the remainder lie either on or above it. We denote this polygon by $\pi(f)$. If n is the rank of a maximum term and $m \neq n$, then plainly

$$g_m - m \log r \geqslant g_n - n \log r.$$

The geometrical interpretation of these inequality is that the points A_m do not lie below the line D_r, of slope(¹) $\log r$, passing through the point A_n. The point A_n is therefore a point of the polygon $\pi(f)$



and the line D_r is a " tangent" to this polygon — it is the tangent of slope $\log r$. A_n is uniquely determined when $\log r$ is not equal to the slope of one of the sides of $\pi(f)$, and for such values of r there is only one term in the series equal to $m(r)$. When $\log r$ is equal to the slope of a side of $\pi(f)$ there are several such terms and their number is equal to the number of the points A_n which lie on this side of the polygon. When more than one term of the series is equal to $m(r)$ we shall agree to regard the term of highest rank amongst them as the *maximum term*. With this convention $N(r)$ will be used

(¹) The slope of a line is the tangent of the angle it makes with the axis of x.

to denote the rank of the maximum term. $N(r)$ is an unbounded, non-decreasing function of r with a left-hand discontinuity wherever r passes through a value such that $\log r$ is equal to the slope of one of the sides of $\pi(f)$, and it can assume all values of n corresponding to those points A_n which are vertices of $\pi(f)$. These values of n we shall call *principal indices*.

Two functions $f_1(z)$ and $f_2(z)$ with the same polygon have the same maximum term. That is to say that the functions $m(r)$ and $N(r)$ are indentical for these two functions. In particular the function

$$W(r) = \sum_{0}^{\infty} e^{-G_n} r^n, \tag{2, 6}$$

where G_n is the ordinate of the point of abscissa n on the curve $\pi(f)$, is a dominant function for $f(z)$ and has the same maximum term. It is the simplest function corresponding to the polygon $\pi(f)$. The ratio

$$R_n = e^{G_n - G_{n-1}} \tag{2, 7}$$

of the coefficients in $W(r)$ corresponding to C_{n-1} and C_n we shall call the *rectified ratio of* C_{n-1} to C_n. The logarithm of R_n is equal to the slope of the side of $\pi(f)$ between the points of abcissae $n - 1$ and n, and is therefore a non-decreasing function of n tending to infinity.

Supposing for simplicity that $|f(0)| = 1$, we have

$$m(r) = \frac{r^{N(r)}}{R_1 R_2 \dots R_{N(R)}} \tag{2, 8}$$

and, since

$$\int_{R_i}^{R_{i+1}} \frac{N(x)}{x}\, dx = i \log \frac{R_{i+1}}{R_i},$$

this may be written

$$\log m(r) = \int_0^r \frac{N(x)}{x}\, dx.$$

In general

$$\text{(2, 9)} \qquad \log m(r) = \log m(r_0) + \int_{r_0}^{r} \frac{N(x)}{x}\,dx$$

for $0 < r_0 < r$.

We can now find a relation between $m(r)$ and $M(r)$. In the first place Cauchy's inequality (1, 6) shews that $m(r)$ is less that $M(r)$. But clearly $M(r)$ does not exceed the value of the function $W(r)$. Suppose that p is an integer greater than $N = N(R)$ and such that the rectified ratio $R_p > r$. Then, for $q \geqslant p$,

$$e^{-G_q} r^q = e^{-G_{p-1}} r^{p-1} \frac{r^{q-p+1}}{R_p \ldots R_q} < m(r) \left(\frac{r}{R_p}\right)^{q-p+1}$$

and hence

$$W(r) = \sum_0^{p-1} e^{-G_n} r^n + \sum_p^{\infty} e^{-G_n} r^n < \left[p + \sum_p^{\infty} \left(\frac{r}{R_p}\right)^{q-p+1} \right] m(r)$$

$$= m(r) \left(p + \frac{r}{R_p - r}\right)$$

In order that the two terms in the bracket may be substantially equivalent we take

$$p = N\left(r + \frac{r}{N(r)}\right) + 1,$$

which implies that

$$R_p > r + \frac{r}{N(r)}.$$

The addition of 1 in this formula for p ensures that in all cases p shall be greater than $N(r)$, in accordance with our hypothesis. Hence we have the following theorem :

THEOREM 11. $$m(r) < M(r) < m(r)\left[2N\left(r + \frac{r}{N(r)}\right) + 1\right] \qquad (2, 10).$$

It can be shewn by means of examples that the coefficient 2 in the bracket cannot be replaced by any number less than 1, and that $N\left(r + \frac{r}{N(r)}\right)$ cannot be replaced by $N(r)$.

It appears in the course of the proof of this theorem that the sum of the remainder of the terms of $W(r)$ after a term of certain rank, in the neighbourhood of $N(r)$, is negligible. It is easy to see that $M(r)$ is asymptotically equivalent to the sum of the first

$$2N\left(r + \frac{r}{N(r)}\right) \log N(r)$$

terms in $f(z)$.

5. Definition of order. Functions of finite order. — It is clear, as is shewn for instance by formula (2, 9), that $m(r)$ increases more rapidly than any power of r. So that in the case of those functions which satisfy the condition

$$(2, 11a) \qquad \frac{\log N(r)}{\log r} < K$$

the relation (2, 10) appears in an especially simple form. In fact the inequalities (2, 10) may be written

$$\log M(r) = \left[1 + \theta \frac{\log [2N(r + r/N(r)) + 1]}{\log m(r)}\right] \log m(r) \qquad (0 < \theta < 1),$$

and we see that

$$(2, 11) \qquad \lim_{r=\infty} \frac{\log M(r)}{\log m(r)} = 1\,;$$

or, in the usual notation,

$$\log M(r) \backsim \log m(r).$$

The functions $\log M(r)$ *and* $\log m(r)$ *are then said to be asymptotically equivalent.*

This is not a general property of functions for which $N(r)$ does not satisfy the condition (2, 11 a) imposed above. We can in fact construct a polygon $\pi(f)$ such that, for an indefinitely increasing sequence of values of r, the number of terms equal to $m(r)$ is actually greater than $m(r)$. It is also sufficiently clear that $N(x)$ may be chosen (and this amounts to choosing $\pi(f)$) so that, though $\log N(r)/\log r$ is always less than a given increasing function, equation (2, 11) is not satisfied by the function $W(r)$. For this reason alone it would be convenient to regard those functions for which $N(r)$ satisfies the condition stated as a class apart. We shall see later on however that they possess a number of other special properties.

To be precise let us suppose that

$$\overline{\lim_{r=\infty}} \frac{\log N(r)}{\log r} = \rho < +\infty.$$

Then, by equation (2, 9),

$$\log m(r) \leqslant \log m(r_1) + \int_{r_1}^{r} x^{\rho+\varepsilon-1}\, dx = \frac{1}{\rho+\varepsilon} r^{\rho+\varepsilon} + k,$$

whence, in virtue of (2, 11),

$$\overline{\lim_{r=\infty}} \frac{\log_2 M(r)}{\log r} = \overline{\lim_{r=\infty}} \frac{\log_2 m(r)}{\log r} \leqslant \rho.$$

Conversely, if we suppose that, beyond a certain value r_1 of r,

$$\log M(r) < r^{\rho_1+\varepsilon}$$

we have, still by equation (2, 9),

$$N(r) \log 2 < \int_{r}^{2r} \frac{N(x)}{x}\, dx < (2r)^{\rho_1+\varepsilon}.$$

So that

$$\overline{\lim_{r=\infty}} \frac{\log N(r)}{\log r} \leqslant \overline{\lim_{r=\infty}} \frac{\log_2 M(r)}{\log r}.$$

The ratios $\dfrac{\log N(r)}{\log r}$ *and* $\dfrac{\log_2 M(r)}{\log r}$ *have therefore the same upper limit as* r *tends to infinity.* The property

$$(2,12) \qquad \overline{\lim_{r=\infty}} \frac{\log_2 M(r)}{\log r} = \rho < +\infty$$

is thus a further characteristic of the class of functions satisfying (2, 11a).

We say that *functions satisfying this condition are integral functions of finite order* ρ. All other integral functions are called *functions of infinite order*. For a function to be of finite order it is plainly necessary (and sufficient) that there should exist a number K such that the inequality

$$\log M(r) < r^K$$

is satisfied beyond a certain value of r. The order is then at most equal to K.

If $f(z)$ is an integral function of order ρ, the rank of its maximum term $m(r)$ is less than $r^{\rho+\varepsilon}$, ε being arbitrarily small, provided that r is greater than r_1. From the inequalities (2, 10) and (2, 1) we deduce

THEOREM 12. — *For a function of finite order* ρ *the inequalities*

$$m(r) < M(r) < m(r) r^{\rho+\varepsilon}$$
$$A(r) < M(r) < A(r) r^{\rho+\varepsilon}$$

are satisfied for $r > r_1$, ε *being arbitrarily small, and, a fortiori*

$$\log m(r) \backsim \log M(r) \backsim \log A(r).$$

Let us now turn to the derivative $f'(z)$ of $f(z)$.
Plainly

$$f(z) = \int_0^z f'(z)dz + f(0),$$

where the integral may be taken along a straight line. So that, choosing z such that $|f(z)| = M(|z|)$, we have, in virtue of the fact that the maximum modulus $M^1(r)$ of $f'(z)$ is an increasing function,

$$M(r) < |f(0)| + rM^1(r).$$

On the other hand, by Cauchy's theorem,

$$f'(x) = \frac{1}{2i\pi}\int_C \frac{f(z)}{(z-x)^2}\,dz,$$

C being the circle $|z-x| = R - r$ $(R > r = |x|)$. Consequently, putting $|f'(x)| = M'(r)$, we have also

$$M'(r) < \frac{1}{R-r}M(R).$$

The functions $M'(r)$ and $M(r)$ therefore satisfy the inequalities

$$(2, 13) \qquad \frac{M(r) - |f(0)|}{r} < M'(r) < \frac{1}{R-r}M(R) \qquad (r < R);$$

inequalities which are valid for functions with a finite radius of convergence as well as for integral functions.

From these inequalities it follows, giving R some special value, say $R = 2r$, that the ratios $\log M(r)/\log r$ and $\log M'(r)/\log r$ have the same limits of indetermination as r tends to infinity. That is to say *the order of a function and the order of its derivative are equal.*

Greater precision is possible in the case of a function of finite order ρ. Let $m'(r)$ be the maximum term and N' the rank of this term in $f'(z)$. Then, for $r > r_1$,

$$M'(r) < m'(r)r^{\rho+\varepsilon}, \qquad m'(r) = N'C_{N'}r^{N'-1} \leqslant N'\frac{m(r)}{r},$$

and so

$$M'(r) < m(r)r^{2\rho-1+\varepsilon} < M(r)r^{2\rho-1+\varepsilon}$$

and we have finally this theorem :

THEOREM 13. — *The relations* (2, 13) *are valid for any function regular in a circle of radius greater than* R, *and in the case of an integral function of finite order* ρ

$$M'(r) < M(r)r^{2\rho-1+\varepsilon}$$

for all sufficiently large values of r *and for an arbitrarily small positive* ε, *and, a fortiori,*

$$\log M'(r) \backsim \log M(r).$$

The order of the derivative is also equal to ρ.

We shall see in Chapter IV how far it is possible to extend these results to cover the case of functions of infinite order. Functions of finite order are, as we have seen, dominated by the maximum term $m(r)$. The logarithm of this term is asymptotically equivalent to the logarithms of $M(r)$, $A(r)$ and $M'(r)$ and its order of magnitude determines the order of magnitude of these functions.

III. — The relation between $M(r)$ and the sequence of coefficients C_n. Functions of finite order and regular growth.

The inequalities (2, 10) enable us in the most general case to calculate an approximate value for $M(r)$ when the moduli C_n of the coefficients in the Taylor series are known. In the particular case of functions of finite order it is shewn by the inequality (2, 11) that an approximation to $\log m(r)$ gives an approximation to $\log M(r)$. The inverse problem, however, is not so easily solved. This problem may be stated as follows : *if we know an approximate value for* $\log M(r)$, *what can we say about the coefficients?* But, first of all we must know what conditions a given function $V(r)$ must satisfy in order that the approximate equation

$$\log M(r) \backsim V(r)$$

may be possible. There is no difficulty in finding a condition in the case of functions of finite order, that is to say the case in which, for all sufficiently large values of r, $V(r) < r^K$. As equation (2, 4) shews, it is necessary that

$$V(r) \backsim \int_{r_0}^{r} \frac{W(x)}{x}\,dx,$$

where $W(x)$ is an increasing function continuous in adjacent intervals. That this condition is also sufficient follows from the fact that integral functions for which $N(r)$ is equal to the integral part of $W(r)$ for all values of r will certainly satisfy the asymptotic equation

$$\log M(r) \backsim \log m(r) \backsim V(r).$$

In particular it is clear that we may take for $V(r)$ any continuous differentiable function subject to the condition that $rV'(r)$ shall be an increasing function : e. g. Ar^{ρ}; $Ar^{\rho}(\log r)^{\rho_1}$; $A(\log r)^{\alpha}$ $(\alpha > 1)$ etc.

If $V(r)$ is such a function we may suppose a fortiori that

$$\overline{\lim_{r=\infty}} \frac{\log M(r)}{V(r)} = 1, \quad \lim_{r=\infty} \frac{\log_2 M(r)}{\log V(r)} = 1, \quad \overline{\lim_{r=\infty}} \frac{\log_2 M(r)}{\log V(r)} = 1.$$

If such an approximation to $M(r)$ is known, then the maximum of the ratio $M(r)/r^n$ provides us with an upper bound for C_n for each value of n, in virtue of Cauchy's inequality

$$C_n r^n < M(r).$$

On the other hand

$$\int_{r_0}^{r} \frac{N(x)}{x} dx < \log M(r) < \int_{r_0}^{r} \frac{N(x)}{x} dx + \log \left[2N\left(r + \frac{r}{N(r)}\right) + 1 \right],$$

where r_0 is some fixed number. From the first of these inequalities we obtain an upper limit for $N(r)$ and substituting this in the term in brackets on the right we obtain from the second inequality a lower limit for the integral, and so for $N(r)$. We have thus found an approximation to $N(r)$, and consequently to the inverse function R_n. Finally the equation

$$m(R_n) = e^{-G_n} R_n^{\ n}$$

gives an approximation to G_n. That is to say that we can find two curves in the plane of (x, y) such that the polygon $\pi(f)$ lies between them.

6. Pratical rules. — In practice we compare the growth of $M(r)$ with that of a known function. Let us suppose that we know an integral function

$$J(r) = \sum_{0}^{\infty} \Gamma_n r^n$$

with positive coefficients such that the ratio Γ_n / Γ_{n-1} is a non-decreas ing function of n, and satisfying the condition (2, 11).

If, for $n > n_0$, the coefficients C_n of another integral function $f(z)$ are less than the coefficients Γ_n, then

$$M(r) \leqslant \sum_{0}^{n_0} C_n r^n + \sum_{n_0+1}^{\infty} \Gamma_n r^n < K r^{n_0} + J(r) < J(r)^{1+\varepsilon} \qquad (\varepsilon > 0).$$

If, for a sequence of numbers $n_1, n_2, \ldots, n_p, \ldots$

$$C_{n_p} > \Gamma_{n_p}$$

then, for

$$\frac{\Gamma_{n_p-1}}{\Gamma_{n_p}} = R_{n_p} \leqslant r < R_{n_{p+1}} = \frac{\Gamma_{n_{p+1}-1}}{\Gamma_{n_{p+1}}}$$

we have

$$M(r) \geqslant C_{n_p} r^{n_p} \geqslant \Gamma_{n_p} R_{n_p}^{n_p} > J(R_{n_p})^{1-\varepsilon},$$

where ε may be taken as small we please provided that r is sufficiently great.

Conversely, suppose that a function $f(z)$ is of finite order or that a preliminary calculation of the upper limit of $N(r)$ has shewn that $f(z)$ satisfies the equality (2, 11). *Then, if for all sufficiently large values of r*

$$M(r) < J(r)^{1-\alpha} \qquad (\alpha > 0),$$

we have ultimately

$$C_n < \Gamma_n.$$

For the foregoing argument has shewn that if the contrary is true for an infinite sequence of values of n this implies that $M(r) > J(r)^{1-\varepsilon}$ for a sequence of values of r, ε being arbitrarily small. But this is contrary to our hypothesis.

If, for a sequence of numbers $r_1, r_2, \ldots, r_p, \ldots$ tending to infinity

$$M(r) > J(r)^{1+\alpha},$$

then for a sequence of values n_q of n

$$C_{n_q} > \Gamma_{n_q}.$$

If, however, $M(r)$ *satisfies this inequality for all sufficiently large values of r, then*

$$-G_n > \log \Gamma_n$$

for all sufficiently large values of n.

The first part of this rule follows from the fact that if, for all sufficiently large values of n, $C_n \leqslant \Gamma_n$, then $M(r) < J(r)^{1+\varepsilon}$.

To prove the second part we observe that if, for a certain value $n > n_0$, $-G_n \leqslant \log \Gamma_n$, then for that value of r for which $N(r) = n$ the maximum term of $f(z)$ will be less than or equal to the corresponding term in $J(r)$, and so less than $J(r)$.

If we know both an upper and a lower limit for $M(r)$ then we know two curves limiting $\pi(f)$ and consequently a lower bound for the coefficients C_n, where n is a principal index.

We proceed to apply these rules, taking as comparison series the function

$$J(r, A, \rho) = \sum_{0}^{\infty} \left(\frac{A}{n^{\frac{1}{\rho}}} \right)^n r^n.$$

Its maximum term for a given value r corresponds to that value of n for which the expression

$$\psi(n) = n\left(\log Ar - \frac{1}{\rho} \log n\right)$$

is a maximum. The derivative of this expression is

$$\psi'(n) = \log Ar - \frac{1}{\rho}\log n - \frac{1}{\rho}$$

and, if $N(r)$ is the rank of the maximum term, the difference between $N(r)$ and the root of the equation $\psi'(n) = 0$ is less than 1. In fact

$$N(r) = \frac{(Ar)^\rho}{e} + \theta \qquad (|\theta| < 1).$$

The function is therefore of finite order equal to ρ and

$$\log J(r, A, \rho) \backsim \frac{1}{\rho e}(Ar)^\rho.$$

THEOREM 14. — *The necessary and sufficient condition that an integral function should be of finite order ρ is that*

$$(2, 14) \qquad \varliminf_{n=\infty} \frac{-\log C_n}{n\log n} \text{ should be equal to } \frac{1}{\rho},$$

The condition is necessary because, if the order is ρ, we have, for sufficiently large values of r,

$$\log M(r) < (1-\alpha) J\left(r, \left[\frac{e(\rho+\varepsilon)}{(1-\alpha)^2}\right]^{\frac{1}{\rho+\varepsilon}}, \rho+\varepsilon\right) \backsim \frac{1}{1-\alpha} r^{\rho+\varepsilon};$$

so that, for all sufficiently large values of n,

$$\sqrt[n]{C_n} < \left[\frac{e(\rho+\varepsilon)}{n(1-\alpha)^2}\right]^{\frac{1}{\rho+\varepsilon}}$$

Hence

$$\varliminf_{n=\infty} \frac{-\log C_n}{n\log n} \geqq \frac{1}{\rho}.$$

On the other hand, for a sequence of values of r,

$$(\beta) \quad \log M(r) > (1+\alpha) J\left(r, \left[\frac{e(\rho-\varepsilon)}{(1+\alpha)^2}\right]^{\frac{1}{\rho-\varepsilon}}, \rho-\varepsilon\right) \backsim \frac{1}{1+\alpha} r^{\rho-\varepsilon};$$

so that for a sequence of values of n

$$(\gamma) \qquad \sqrt[n]{C_n} > \left[\frac{e(\rho-\varepsilon)}{n(1+\alpha)^{\rho}}\right]^{\frac{1}{\rho-\varepsilon}}$$

and

$$\overline{\lim_{n=\infty}} \frac{-\log C_n}{n \log n} \leqslant \frac{1}{\rho}.$$

It can also be shewn without difficulty that the condition is sufficient.

The use of the comparison function $J(r, A, \rho)$ enables us to prove in a similar manner that :

If ρ, B and D are given numbers, and $B \leqslant D$, the necessary and sufficient condition that a function should satisfy the inequalities

$$B \leqslant \overline{\lim_{r=\infty}} \frac{\log M(r)}{r^{\rho}} \leqslant D$$

is that we should have

$$B \leqslant \overline{\lim} \left[\frac{n}{e\rho} C_n^{\rho/n}\right] \leqslant D.$$

The use of more complicated comparison functions, such for instance as make $\log M(r)$ approximate to $Ar^{\rho}(\log r)^{\rho_1}$ etc., leads to analogous results.

7. Functions of regular growth. — In this section we have so far restricted ourselves to the consideration of what *Borel has called functions of regular growth.* That is to say such that,

$$(2, 15) \qquad \lim_{n=\infty} \frac{\log_2 M(r)}{\log r} = \rho.$$

The inequality

$$\log M(r) > r^{\rho-\varepsilon}$$

is then true for all sufficiently large values of r, ε being arbitrarily chosen. For such functions therefore

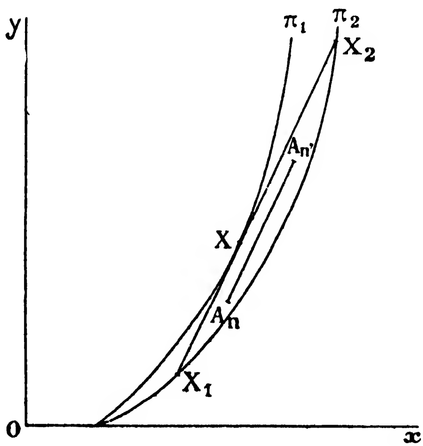
$$\lim_{n=\infty} \frac{G_n}{n \log n} = \frac{1}{\rho}.$$

For the inequality (β) is now valid for sufficiently large values of r, and so (γ), where C_n is replaced by e^{-G_n}, is valid for all sufficiently large values of n. For all n greater than a certain value the polygon $\pi(f)$ will lie between the two curves

$$\pi_1 \qquad y = \frac{1}{\rho - \varepsilon} x \log x,$$

$$\pi_2 \qquad y = \frac{1}{\rho + \varepsilon} x \log x.$$

The polygon $\pi(f)$ is convex downwards. If n and n' are two consecutive principal indices, that is to say that A_n, $A_{n'}$, are two consecutive vertices of $\pi(f)$, the chord $A_n A_{n'}$ must lie between the two curves π_1 and π_2. The interval between the numbers n and n' is a maximum when A_n and $A_{n'}$ are the points of intersection with π_2 of a tangent to π_1. Let X be a point of π_1 and X_1, X_2 the points of inter-



section with π_2 of the tangent at $X(X_1 < X, X_2 > X)$. X_1 and X_2 are the roots of the equation

$$\frac{\rho - \varepsilon}{\rho + \varepsilon} x \log x - X \log X = (\log X + 1)(x - X)$$

or

$$x\left(\frac{\rho - \varepsilon}{\rho + \varepsilon} \log x - \log X - 1\right) + X = 0.$$

The left-hand side is positive for $x = X/(\log X + 1)$. Therefore, since the expression is negative for $x = X$, X_1 is greater than this number $X/(\log X + 1)$. Similarly the right-hand side is positive for $\log x = \frac{\rho + \varepsilon}{\rho - \varepsilon}(\log X + 1)$, so that $\log X_2$ is less than this expression, and we have

$$\frac{\log X_2}{\log X_1} < \frac{\rho + 2\varepsilon}{\rho - \varepsilon} \qquad (X_1 > X_0).$$

Hence

$$\frac{\log n'}{\log n} < \frac{\rho + 2\varepsilon}{\rho - \varepsilon}.$$

The coefficients C_n are therefore equal to the numbers e^{-G_n} for a sequence of values of n such that the ratio of the logarithms of two consecutive values of n in the sequence is less than $\frac{\rho + 2\varepsilon}{\rho - \varepsilon}$. But ε is arbitrary and we see that there is a sequence of numbers $n_1, n_2, \ldots, n_p, \ldots$ such that

$$\lim_{p=\infty} \frac{\log n_{p+1}}{\log n_p} = 1,$$

and

$$\sqrt[n_p]{C_{n_p}} > n_p^{-\frac{1}{\rho - \varepsilon_p}}, \qquad \varepsilon_p \to 0 \quad \text{as} \quad p \to \infty.$$

This condition, in conjunction with (2, 14), is sufficient to ensure that the function shall be of regular growth. For if it is fulfilled

the polygon $\pi(f)$ lies entirely below the curve π_2 for $n > n_1$, and, it follows that $M(r)$ is greater than the maximum term of the function $J(r, 1, \rho - \varepsilon)$. Hence we have the following theorem.

THEOREM 15. — *The necessary and sufficient condition that a function of order ρ should be of regular growth is that the coefficients C_n should satisfy the inequality*

$$\sqrt[n]{C_n} < n^{-\frac{1}{\rho+\varepsilon}},$$

ε being arbitrarily small and positive, for all sufficiently large values of n, and that there should be an infinite sequence of numbers n_p, such that

$$\lim_{p=\infty} \frac{\log n_{p+1}}{\log n_p} = 1,$$

for which

$$\sqrt[n_p]{C_{n_p}} > n_p^{-\frac{1}{\rho-\varepsilon_p}}, \qquad \textit{where} \quad \varepsilon_p \to 0 \quad \textit{as} \quad p \to \infty.$$

By similar methods it is possible to obtain more precise results, such as the following :

The necessary and sufficient condition for

$$(2, 16) \qquad \lim_{r=\infty} \frac{\log M(r)}{r^\rho} = B$$

is that, for all values of ε and all sufficiently large values of n, we should have

$$\frac{1}{\rho e} n C_n^{\rho/n} < B + \varepsilon,$$

and that there should exist a sequence of numbers n_p, such that

$$\lim_{p=\infty} (n_{p+1}/n_p) = 1,$$

for which

$$\lim_{p=\infty} \left[\frac{1}{\rho e} n_p C_{n_p}^{\rho/n_p}\right] = B.$$

Functions which satisfy the condition (2, 16) we shall call *functions of perfectly regular growth.* In the sequel we shall consider an extensive class of differential equations and we shall shew that integral functions which satisfy these equations are of this kind. A function of order ρ will be said to be of *very regular growth* when two finite positive numbers B and D can be found such that

$$(2, 17) \qquad \varliminf_{r=\infty} \frac{\log M(r)}{r^\rho} \geqslant B, \qquad \varlimsup_{r=\infty} \frac{\log M(r)}{r^\rho} \leqslant D;$$

The same method suffices to shew that the necessary and sufficient condition for the existence of such numbers is that, beyond a certain value of n, the polygon $\pi(f)$ should lie between the two curves

$$y = \frac{x}{\rho} \log \frac{x}{(B-\varepsilon)e\rho}, \qquad y = \frac{x}{\rho} \log \frac{x}{(D+\varepsilon)e\rho}.$$

But we are no longer able to find necessary and sufficient conditions of the same form as in the preceding cases, where we were able to approximate to $\log M(r)$ as closely as we pleased.

In the present case it can be shewn that

$$\varlimsup_{n=\infty} \left(\frac{1}{\rho e} n C_n^{\rho/n}\right) \leqslant D;$$

and that

$$\varliminf_{p=\infty} \left(\frac{1}{\rho e} n_p C_{n_p}^{\rho/n_p}\right) \geqslant B$$

for a sequence of numbers n_p satisfying the condition

$$\varlimsup_{p=\infty} \frac{n_{p+1}}{n_p} \leqslant \frac{x_1}{x_2},$$

where x_1 is the greatest and x_2 the smallest root of the equation

$$x \log \frac{x}{e} + \frac{B}{D} = 0.$$

But these conditions, while they imply the second of the relations (2, 17), give a less precise result for the lower limit. In fact we can only deduce

$$\varliminf_{r=\infty} \frac{\log M(r)}{r^\rho} \geqslant B\frac{B}{D}.$$

The oscillation of the polygon $\pi(f)$ between the two curves depends on the nature of the oscillation of $\log M(r)$ between Br^ρ and Dr^ρ and of this we are in ignorance.

This example reveals, perhaps better than the previous ones, the interest attaching to Hadamard's polygon $\pi(f)$.

8. Functions defined by Poincaré's functional equations[1]. — Under certains conditions, which it is unnecessary to discuss here the functions defined by the equation

$$(2, 18) \qquad f(sz) = P_0(z)[f(z)]^p + \ldots + P_p(z) \quad (S = |s| > 1),$$

where $P_j(z)$ is a polynomial, are integral functions. They are of very regular growth, and it is as examples of such functions that we consider them here. Let $M(r)$ be the maximum modulus of a solution of (2, 18). To calculate this function put

$$P_0(z) = a_0 z^q + \ldots \qquad (A = |a_0|)$$

and we have plainly

$$M(Sr) = (1 + h(r)/r)Ar^q M(r)^p, \qquad (h(r) \text{ bounded})$$

since $M(r)$ ultimately surpasses any number of the form r^K. If we suppose that $p > 1$ it follows without difficulty by repeated applications of this relation that

$$M(S^n r_0) = [(1 + \eta(n))\, K(r_0)]^{p^n} \qquad \left(\lim_{n=\infty} \eta(n) = 0\right).$$

---

[1] Poincaré, *Journal de Mathématiques*, 1890.

Consequently

$$\log M(r) \backsim H(\log r)\, r^{\log p/\log S}$$

$H(x)$ being a periodic function and its period $\log S$. We have thus a class of functions of very regular growth of order $\log p/\log S$.

When $p = 1$, we obtain, as a result of repeated applications of the first relation

$$M(S^n r_0) = H(n) M(r_0) (A r_0^{\,q})^n S^{q\frac{n(n-1)}{2}},$$

where $H(n)$ lies between two positive numbers. Therefore

$$\log M(r) \backsim \frac{q}{2\log S}(\log r)^2$$

and the function is found to be *of zero order* and *perfectly regular growth.* The properties of its coefficients could be discussed by the general method of paragraph 6.

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CHAPTER III.

The zeros of functions of finite order and Borel's theorem.

The theory of integral functions, or more generally of the functions $F(z)$ having an isolated singularity at infinity, may be developed in two directions. On the one hand we may seek to deduce from facts about the zeros information concerning the formal factorisation of $F(z)$. On the other hand, regarding the problem from the point of view of the theorems of Weierstrass and Picard, we may endeavour to acquire a deeper insight into the nature of the function by investigating the properties of the roots of the equations $F(z) - a = 0$. The study of the zeros of these functions thus serves a double purpose, since it contributes to advance the theory along both these avenues. In this chapter we give first of all the theorems due to Hadamard and Borel concerning the formal factorisation of $F(z)$ and then proceed to a direct investigation of the moduli of its zeros by the methods of Borel. The results that we shall prove bring out very clearly the close relationship existing between the two points of view.

I. — The exponent of convergence and the formal factorisation.

1. Jensen's theorem. — The following theorem, which is due to Jensen, is fundamental in our treatment of the subject.

Theorem 16. — *Let $f(z)$ be an integral function such that $f(0) \neq 0$ and let $r_1, r_2, \ldots, r_n, \ldots$ be the moduli of its zeros arranged as a non-decreasing sequence. Then, if $r_n \leqslant r \leqslant r_{n+1}$,*

$$(3, 1) \qquad \log \frac{r^n |f(0)|}{r_1 r_2 \ldots r_n} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\varphi})| \, d\varphi .$$

Both sides of this equation have a derivative which is continuous except at the points $r = r_n$. This is obvious in the case of the left hand side; and, since $f'(re^{i\varphi})/f(re^{i\varphi})$ is uniformly continuous in r and φ in any domain which does not contain a line $r = r_n$, the same is true of the integral also. Further these two derivatives are equal, for if r lies between r_n and r_{n+1} we have, by Cauchy's theorem,

$$n = \frac{1}{2i\pi}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\int_0^{2\pi} \frac{f'(re^{i\varphi})}{f(re^{i\varphi})}\,re^{i\varphi}\,d\varphi,$$

which may be written

$$\frac{n}{r} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr}[\log f(re^{i\varphi})]\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{dr}\log|f(re^{i\varphi})|\,d\varphi$$

$$= \frac{d}{dr}\left[\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\varphi})|\right]d\varphi,$$

and the first and last of these expressions are the derivatives of the left and right-hand sides of equation (3, 1) respectively. Therefore, since the two expressions in equation (3, 1) are equal when $r = 0$, it will be sufficient to prove that they are continuous to establish the theorem. In the case of the left-hand side this is obvious. Now consider the integral on the right. $\operatorname{Log}|f(re^{i\varphi})|$ is continuous in r and φ except for $r = r_n$ and, if $\varphi_1, \varphi_2, \ldots, \varphi_p$ are the arguments of the zeros of modulus r_n, we may write

$$\log|f(re^{i\varphi})| = \sum_1^p \log\left|1 - \frac{r}{r_n}e^{i(\varphi-\varphi_q)}\right| + \psi(r, \varphi),$$

where the function $\psi(r, \varphi)$ is continuous in a rectangle

$$r_n - h \leqslant r \leqslant r_n + h.$$

The problem is thus reduced to that of shewing that the integral

$$\int_0^{2\pi} \log\left|1 - \frac{r}{r_n}e^{i(\varphi-\varphi_q)}\right| d\varphi$$

exists and is continuous for $r = r_n$. This follows from the inequalities

$$\left|1 - \frac{r}{r_n} e^{i(\varphi-\varphi_q)}\right| = \left|1 + \left(\frac{r}{r_n}\right)^2 - 2\frac{r}{r_n}\cos(\varphi - \varphi_q)\right|^{\frac{1}{2}}$$
$$\geqslant \sin|\varphi - \varphi_q| \geqslant |\varphi - \varphi_q| \frac{2}{\pi};$$

$$|1 - e^{i(\varphi-\varphi_q)}| = 2\sin\left|\frac{\varphi - \varphi_q}{2}\right| \leqslant |\varphi - \varphi_q|;$$

$$\left|1 - \frac{r}{r_n} e^{i(\varphi-\varphi_q)}\right| < 3 \quad \text{if} \quad r - 2r_n,$$

valid for $|\varphi - \varphi_q| < \frac{\pi}{2}$. For the first two shew that the integral exists for $r = r_n$, and the first and third shew that, provided $r < 2r_n$,

$$2\delta \log \frac{\pi e}{2\delta} < \left|\int_{-\delta+\varphi_q}^{\delta+\varphi_q} \log\left|1 - \frac{r}{r_n} e^{i(\varphi-\varphi_q)}\right| d\varphi\right| < 2\delta \log 3$$

which implies continuity.

As a deduction from Jensen's theorem we have the inequality

$$\frac{r^n}{r_1 r_2 \ldots r_n} |f(0)| < M(r) \qquad (r_n \leqslant r \leqslant r_{n+1}),$$

which is strictly analogous to Cauchy's inequality (1, 6) and can be put to similar use. For, since the function $\dfrac{r^m}{r_1 \ldots r_m}$ is a maximum when $m = n$, the inequality holds for all values of n and r. If $f(0) = 0$ and c_p is the first coefficient which is not zero the inequality can be applied to the function $f(z)/z^p$.

As in the case of an analogous expression (equation (2, 9)) $\log \dfrac{r^n}{r_1 r_2 \ldots r_n}$ can be transformed into an integral. Let $n(x)$ denote the number of zeros, at the origin or elsewhere, in the circle $|z| \leqslant x$. Then

$$\log \frac{r^n}{r_1 r_2 \ldots r_n} = \int_{r_1}^{r} \frac{n(x) - p}{x} dx,$$

and so generally

(3, 2) $$\int_{r_1}^{r} \frac{n(x)}{x}\,dx < \log M(r) - \log |c_p| - p \log r_1.$$

This last inequality shews that from the value of $\log M(r)$, we can deduce an upper limit for the number of zeros of the function.

2. The sequence of zeros and the exponent of convergence. — Let $M(r)$ be the maximum modulus of a function of finite order ρ. Then, given ε,

$$\log M(r) < r^{\rho+\varepsilon} \qquad (r > r_\varepsilon).$$

Therefore, by (3, 2),

$$n(r) \log 2 < \int_{r}^{2r} \frac{n(x)}{x}\,dx < (2r)^{\rho+\varepsilon} + K\,(^1),$$

and so

(3, 3) $$\rho_1 = \overline{\lim_{n=\infty}} \frac{\log n}{\log r_n} = \overline{\lim_{r=\infty}} \frac{\log n(r)}{\log r} \leqslant \rho.$$

The number ρ_1 defined by this equation has the following important property : *The series*

(3, 4) $$\sum_{1}^{\infty} \frac{1}{r_n^{\tau}}$$

is convergent for $\tau > \rho_1$ *and divergent for* $\tau < \rho_1$.

We must clearly suppose that ρ_1 is finite. To prove the first part of the proposition we observe that, by the definition of ρ_1, $n < r_n^{\rho_1+\varepsilon}$ for all sufficiently large values of n; and, τ being greater than ρ, we can choose ε so that $\rho_1 + \varepsilon < \tau$ Consequently

$$\frac{1}{r_n^{\tau}} < \frac{1}{n^{\tau_1}} \qquad \left(\tau_1 = \frac{\tau}{\rho_1 + \varepsilon} > 1\right),$$

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(1) To avoid repetitions, we shall denote by K or by K_i any positive constant which does not depend on the variables.

provided that n is sufficiently large, and the convergence of the series is established.

If $\tau < \rho_1$ we have, for a sequence of values of n,

$$u_n = \frac{1}{r_n^\tau} > \frac{1}{n}.$$

Since the numbers u_n form a non-decreasing sequence it follows from this inequality, putting m equal to the integral part of $n/2$, that

$$u_m + u_{m+1} + \dots + u_n > \frac{n-m}{n} > \frac{1}{2}.$$

In this case the series is therefore divergent.

Now observe that

$$\sum_1^m \frac{1}{r_n^\tau} = \tau \int_{r_1}^{r_{m+1}} \frac{n(x)}{x^{\tau+1}} dx + \frac{m}{r_{m+1}^\tau}.$$

So if the series is convergent the integral on the right is convergent a fortiori. Conversely if the integral is convergent, $n(x)/x^\tau$ tends to zero, for we have

$$\tau \int_r^{2r} \frac{n(x)}{x^{\tau+1}} dx > \frac{n(r)}{r^\tau}(1 - 2^{-\tau}).$$

The right-hand side of the equation is therefore bounded and the series is also convergent. We have then this further property of the series (3, 4) : *The series* (3, 4) *and the integral*

(3, 5) $$\int_0^\infty \frac{n(x)}{x^{\tau+1}} dx$$

converge and diverge together.

The number ρ_1 is called *the exponent of convergence of the sequence* r_n. The following theorem is simply a statement in different terms of inequality (3, 3) :

THEOREM 17. — *The exponent of convergence of the sequence of the zeros of a function of finite order* ρ *does not exceed* ρ.

3. Canonical products. — Let ρ_1 be the exponent of convergence of the sequence of zeros of a function of finite order. Availing ourselves of the remark at the end of § 1.6 we can construct a Weierstrassian product with these zeros. It is apparent that matters will be simplified by taking p as small as possible. There are then two cases to consider according as ρ_1 is or is not an integer :

(i) when ρ_1 is not an integer we take p equal to the integral part of ρ_1.

(ii) when ρ_1 is an integer we take $p = \rho_1$ if the series (3, 4) diverges for $\tau = \rho_1$, and $p = \rho_1 - 1$ if the series converges for $\tau = \rho_1$.

In other words p will be the smallest integer such that the series (3, 4) is convergent for $\tau = p + 1$. The product

$$P(z) = \prod_1^\infty E\left(\frac{z}{a_n}, p\right)$$

we shall call *the canonical product* for the given zeros, p being its *genus*. Borel has proved two fundamental theorems concerning canonical products which we shall deduce from the following inequalities :

If $P(z)$ *is a canonical product of genus* p *and* k *a positive number greater than* 1, *then*

$$\log|P(z)| < KI \tag{3, 6}$$

and

$$\log|P(z)| > -KI + \log\prod_1^N\left|1 - \frac{z}{a_n}\right| \qquad (r_N \leqslant kr < r_{N+1}), \tag{3, 7}$$

where

$$I = \int_0^\infty \frac{n(x)}{x^{p+1}}\frac{r^{p+1}}{x+r}dx.$$

These inequalities can be deduced from those obtained in the proof of theorem 6. The number N being defined as above we had (inequalities (1, 9) and (1, 10))

$$P(z) = P_N(z)Q(z),$$

where

$$\left|\log|Q(z)|\right| < \frac{k}{k-1}\sum_{N+1}^{\infty} u_n^{p+1} = V \qquad \left(u_n = \frac{r}{r_n}\right).$$

Now, for $|u| \geqslant \frac{1}{k}$ and all values of p,

$$\begin{aligned}\log|E(u,p)| &\leqslant \log(1+|u|) + |u| + \dots + \frac{|u|^p}{p}\\ &< \log|u| + \log(1+k) + |u|^p\left(\frac{1}{p} + \frac{k}{p-1} + \dots + k^p\right)\\ &< \log|u| + K_1 + K_2|u|^p < \log|u| + K_3\frac{|u|^{p+1}}{1+|u|},\end{aligned}$$

and

$$\begin{aligned}\log|E(u,p)| &> \log|1-u| - \left(|u| + \dots + \frac{|u|^p}{p}\right)\\ &> \log|1-u| - \log|u| - K_3\frac{|u|^{p+1}}{1+|u|}.\end{aligned}$$

Therefore, writing

$$U = K_3\sum_{1}^{N}\frac{u_n^{p+1}}{1+u_n} + \log\frac{r^N}{r_1 r_2 \dots r_N},$$

we have

$$\log\prod_{1}^{N}\left|1-\frac{z}{a_n}\right| - U < \log|P_N(z)| < U.$$

But

$$V < K_4\sum_{N+1}^{\infty}\frac{u_n^{p+1}}{1+u_n},$$

so that, replacing the logarithm by an integral, we have

$$U + V < K_5\sum_{1}^{\infty}\frac{u_n^{p+1}}{1+u_n} + \int_0^r \frac{n(x)}{x}\,dx.$$

Now the series on the right can also be transformed into an integral, for

$$\int_{r_n}^{r_{n+1}} n(x)\,d\left(\frac{-\left(\frac{r}{x}\right)^{p+1}}{1+\frac{r}{x}}\right) = n\left(\frac{u_n^{p+1}}{1+u_n} - \frac{u_{n+1}^{p+1}}{1+u_{n+1}}\right)$$

and, since the series of positive terms $\sum \frac{1}{r_n^{p+1}}$ is by hypothesis convergent, the expression

$$n\,\frac{u_{n+1}^{p+1}}{1+u_{n+1}} = \frac{r^{p+1}}{1+u_{n+1}}\,\frac{n}{r_{n+1}^{p+1}}$$

ends to zero with $1/n$. Hence

$$\sum_1^{\infty} \frac{u_n^{p+1}}{1+u_n} = \int_0^{\infty} \frac{n(x)r^{p+1}}{x^{p+1}(x+r)}\,\frac{pr+(p+1)x}{x+r}\,dx.$$

The factor $(pr+(p+1)x)/(x+r)$ in the integrand is confined between the limits p and $p+1$ and may therefore be replaced in the inequality by constant factor $p+1$. Thus we obtain

$$U + V < \int_0^r \frac{n(x)}{x}\,dx + K_0 I.$$

Finally the coefficient of $\frac{n(x)}{x}dx$ in the integral I is always greater than $\frac{1}{2}$ in the range $(0, r)$, so that the first integral in this inequality is always less than $2\,I$ and consequently

$$U + V < KI,$$

from which the inequalities (3, 6), and (3, 7) follow.

Consider a function of exponent ρ_1. If ρ_1 is less than $p+1$ we can choose ε such that $\rho_1 + \varepsilon < p + 1$ and

$$n(x) < x^{\rho_1+\varepsilon}$$ 36237

for $x > x_1$. Substituting this in (3, 6) we have

$$\log M(r) < K_1 + K r^{p+1} \int_{r_1}^{\infty} \frac{x^{\rho_1+\varepsilon-p-1}}{x+r}\,dx$$

$$< K_1 + K r^{p} \int_{r_1}^{r} x^{\rho_1+\varepsilon-p-1}\,dx + K r^{p+1} \int_{r}^{\infty} x^{\rho_1+\varepsilon-p-2}\,dx$$

$$= K_1 + K_2 r^{\rho_1+\varepsilon},$$

since $(x + r)$ is greater than r and x. The order of $P(z)$ does not therefore exceed ρ_1.

If $\rho_1 = p + 1$ the integral (3, 5) is convergent for $\tau = p + 1$. So, writing

$$\log M(r) < K r^{p+1} \int_0^{r/H} \frac{n(x)}{x^{p+1}(x+r)}\,dx + K r^{p+1} \int_{r/H}^{\infty} \frac{n(x)}{x^{p+2}}\,dx,$$

we see that the second of these integrals may be made arbitrarily small by choosing r/H sufficiently large and that the first is less than

$$\frac{1}{1+H} \int_0^{\infty} \frac{n(x)}{x^{p+2}}\,dx.$$

Since H may be chosen as large as we please it follows that

$$(3,\ 8) \qquad \lim_{r=\infty} \frac{\log M(r)}{r^{\rho_1}} = 0,$$

and in this case also the order of $P(z)$ does not exceed ρ_1. Comparing this result with theorem 17 we obtain the first of the two theorems in view :

Theorem 18. — *The order of a canonical product is equal to the exponent of convergence of the sequence of its zeros.*

From this theorem in conjunction with inequality (3, 7) we can deduce the second of Borel's results :

THEOREM 19. — *If about each zero a_n of modulus greater than 1 as centre there is described a circle of radius $r_n^{-h}(h > \rho_1)$, then in the domain excluded from these circles*

$$\log |P(z)| > -r^{\rho_1+\epsilon'}, \tag{3, 9}$$

provided that $r > r_{\epsilon'}$.

In the first place the series of the radii of the circles described about the zeros is convergent, so that they certainly do not cover the whole plane, and the excluded domain actually exists. Moreover it is clear that in any annulus $R, R+\lambda$ of sufficiently large radius there are circles $|z| = r$ which do not cut the small circles about the zeros. This is the main property utilised in the sequel.

Now, recalling the argument of theorem 18, we have, in virtue of inequality (3, 7), for $r > r_\epsilon$,

$$\log |P(z)| > -r^{\rho_1+\epsilon} + \log \prod_1^N \left|1 - \frac{z}{a_n}\right|, \qquad (N = n(kr)).$$

For those zeros which lie inside the unit circle, and for all sufficiently large values of r, $\left|1 - \frac{z}{a_n}\right| > 1$. If z lies outside the excluding circles we have for all the other zeros

$$\left|1 - \frac{z}{a_n}\right| > \frac{|z - a_n|}{kr} > \frac{1}{r_n^h kr} > (kr)^{-h-1}.$$

Therefore, for all sufficiently large values of r,

$$\log \prod_1^N \left|1 - \frac{z}{a_n}\right| > -(h+1)\log(kr)\,.\,n(kr) > -K \log r\,.\,r^{\rho_1+\epsilon}$$
$$> -r^{\rho_1+2\epsilon},$$

ϵ being an arbitrarily assigned positive number. This proves the theorem.

In particular it is shewn that there is a sequence of circles of inde-

finitely increasing radii on which the maximum modulus of $1/P(z)$ is of the same order of magnitude as the maximum modulus of $P(z)$. As we shall see later on this is a general property of integral functions.

4. Formal factorisation. — We are now in a position to prove Hadamard's fundamental theorem concerning the factorisation of a function $F(z)$ having an isolated essential singularity at infinity.

THEOREM 20. — *A function* $F(z)$ *of finite order* ρ *is of the form*

$$F(z) = z^{\alpha}\Phi\left(\frac{1}{z}\right) e^{Q(z)} P(z),$$

where $Q(z)$ *is a polynomial of degree* q, $P(z)$ *a canonical product of order* ρ_1,

$$\rho_1 \leqslant \rho, \qquad q \leqslant \rho$$

and ρ *does not exceed the greater of the numbers* ρ_1 *and* q.

Since $F(z) = f(z)\ \Phi\left(\frac{1}{z}\right)$, where $f(z)$ is an integral function of order ρ, the sequence of zeros of $F(z)$ will have an exponent of convergence ρ_1 not exceeding ρ. Therefore, if $P(z)$ is the canonical product formed with these zeros, we have (1, 11)

$$e^{g(z)} = \frac{F(z)}{z^{\alpha}\Phi\left(\frac{1}{z}\right) P(z)},$$

where $g(z)$ is an integral function. It follows from theorem 19 and the definition of order that on a sequence of circles of indefinitely increasing radii

$$\left|e^{g(z)}\right| < r^{\kappa}\, e^{r^{\rho+\varepsilon}}\, e^{r^{\rho_1+\varepsilon}} < e^{r^{\rho+2\varepsilon}}.$$

The real part of $g(z)$ is therefore less than $r^{\rho+2\varepsilon}$ on this sequence of circles, and so, by corollary 7(*b*), $g(z)$ is a polynomial of degree q

less than or equal to $\rho + 2\varepsilon$. Since ε is arbitrary this number may be replaced by ρ and, seeing that the order of $e^{q(z)}$ is is equal to q, the proposition is proved.

COROLLARY 20. — *If* F(z) *is of non-integral order* ρ, *the exponent of convergence* ρ_1 *is equal to* ρ.

For, since q is then certainly less than ρ_1, $\rho_1 = \rho$.

The *genus of* F(z) is defined to be the greatest of the numbers p (genus of P(z)) and q. *The genus of a function of non-integral order is therefore equal to the integral part of the order.* In this case the form of the factorisation is completely determinate, and for an integral function whose Taylor series is known the actual coefficients in Q(z) can be found independently of any knowledge of the zeros. To prove this we suppose for simplicity that $f(0) = 1$. Then

$$f(z) = 1 + c_1 z + \ldots + c_p z^p + \ldots = e^{b_1 z + \cdot + b_p z^p} \mathrm{P}(z).$$

Now, for $|u| < 1$,

$$\mathrm{E}(u, p) = e^{-\frac{u^{p+1}}{p+1} - } = 1 - \frac{u^{p+1}}{p+1} - \ldots.$$

Consequently

$$\mathrm{P}(z) = 1 - \frac{z^{p+1}}{p+1} \sum_{1}^{\infty} \frac{1}{a_n^{p+1}} \cdots$$

and $b_1, b_2, \ldots, b_p$ can be calculated by equating coefficients.

On the other hand when the order ρ is an integer there are five possibilities :

(i)	$\rho_1 < \rho$,	$p \leqslant \rho_1$,	$q = \rho$	genus is equal to ρ,
(ii)	$\rho_1 = \rho$,	$p = \rho$,	$q = \rho$	» » » » ρ,
(iii)	$\rho_1 = \rho$,	$p = \rho$,	$q < \rho$	» » » » ρ,
(iv)	$\rho_1 = \rho$,	$p = \rho - 1$	$q = \rho$	» » » » ρ,
(v)	$\rho_1 = \rho$,	$p = \rho - 1$	$q < \rho$	» » » » $\rho - 1$.

In case (v) the modulus of the canonical product satisfies condition (3, 8) and, since $Q(z)$ is of degree less than ρ,

$$(3, 10) \qquad \lim_{r=\infty} \frac{\log M_1(r)}{r^{\rho}} = 0.$$

Similarly in cases (i) and (iv)

$$\overline{\lim_{r=\infty}} \frac{\log M_1(r)}{r^{\rho}} < +\infty.$$

Consequently if

$$(3, 11) \qquad \overline{\lim_{r=\infty}} \frac{\log M_1(r)}{r^{\rho}} = +\infty.$$

the genus, exponent of convergence and order are equal and the series (3, 4) is divergent for $\tau = \rho$; if

$$(3, 12) \qquad \overline{\lim_{r=\infty}} \frac{\log M_1(r)}{r^{\rho}} > 0$$

the genus and the order are equal.

We shall see later that when condition (3, 10) is satisfied the genus may depend on the first coefficient of the function, and therefore cannot always be determined by the asymptotic behaviour of $M_1(r)$.

In the case of functions of integral order the terms of the polynomial $Q(z)$ can be found if the zeros are known or if the series (3, 4) is divergent for $\tau = \rho$.

5. Examples. — To illustrate the results of the preceding paragraph we give below two examples of their application to special functions.

Consider first the function $\sin(\pi z)$. It is shewn by Euler's formula

$$\sin(\pi z) = \frac{1}{2i}(e^{i\pi z} - e^{-i\pi z})$$

that the only zeros of the function are at the points $0, \pm 1, \pm 2, \ldots$ and that $\log M(r) < \pi r$. The order, exponent of convergence and

consequently the genus of $\sin \pi z$ are therefore equal to 1, and we have

$$\sin \pi z = \pi z e^{bz} \prod_1^\infty \left(1 - \frac{z}{n}\right) e^{\frac{z}{n}} \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

or, expanding both sides,

$$\pi z - \frac{\pi^3 z^3}{6} + \ldots = \pi z(1 + bz + \ldots)(1 + \alpha z^2 + \ldots).$$

Therefore $b = 0$, and we obtain the familiar result

$$\sin \pi z = \pi z \prod_1^\infty \left(1 - \frac{z^2}{n^2}\right).$$

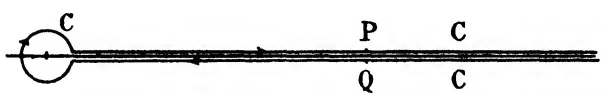
As our second example we consider the function $\Gamma(z)$ defined, for $\Re z > 0$ ($\Re z$ = real part of z), by the integral

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx.$$

Let $\Phi(z)$ be the function defined by the integral

$$\Phi(z) = \int_C e^{-u} u^{z-1} du,$$

where the contour C is formed by the two edges of a cut along the positive real axis and a small circle about the origin in the plane of u.



The integration is effected in the direction of the arrows and we take $u^{z-1} = e^{(z-1)\log u}$, the logarithm being real at the upper edge of the cut along the real axis.

$\Phi(z)$ is an integral function of z, for it is clearly defined for all values of z and is easily seen to be continuous and differentiable. To determine its order consider two points P and Q of abscissa X on the contour. If c is the radius of the circle about the origin we have, putting $|z| = r$,

$$\left|\int_{\mathrm{QOP}} e^{-u} u^{z-1}\, du\right| < 2\mathrm{X} c^{-c} e^{2\pi r} \mathrm{X}^{r-1} + 2\pi c e^{c} e^{2\pi r} c^{r-1}.$$

The remainder of the integral is in absolute value less than

$$2e^{2\pi r} \int_{\mathrm{X}}^{\infty} e^{-u} u^{r-1}\, du \qquad\qquad (u \text{ real})$$

and so, if $\mathrm{X}^{r-1} = e^{\mathrm{X}/2}$, less than $4e^{2\pi r}$. Therefore, if r is sufficiently large, $\mathrm{X} = \mathrm{K} r \log r$ and

$$\log |\Phi(z)| < \mathrm{K} r \log r,$$

thus shewing that the order of $\Phi(z)$ is not greater than 1.

Assuming $\Re z > 0$ and letting $c \to 0$ we find

$$(\alpha) \qquad \Phi(z) = (1 - e^{2i\pi z})\Gamma(z) = -2i \sin(\pi z)\, e^{i\pi z}\Gamma(z),$$

and it follows from this equation that $\Gamma(z)$ can be continued over the whole plane and that its only singularities are poles. The equation

$$(\beta) \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

which is easily established for real values of z between 0 and 1, is therefore valid over the whole plane. Substituting this value for $\sin \pi z$ in equation (α) we have

$$\frac{2i\pi}{\Gamma(1-z)} = -\, e^{-i\pi z}\Phi(z)$$

The function $\dfrac{1}{\Gamma(1-z)}$, and so $\dfrac{1}{\Gamma(z)}$, is thus an integral function of order not greater than 1. Further, it follows from equation (α) that

the zeros of $\frac{1}{\Gamma(z)}$ coincide with certain of the zeros of sin πz. But, since $\Gamma(n) = (n-1)!$ for positive integral values of n, the only possible zeros are the points $0, -1, -2, \ldots$ and it is apparent from equation (β) that these points are actual zeros of $\frac{1}{\Gamma(z)}$, which is therefore an integral function of genus 1. Thus

$$\frac{1}{\Gamma(z)} = Ae^{bz} z \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} = A e^{bz} z\, P(z).$$

In virtue of (β) we have

$$\lim_{z=0} \left(\frac{1}{z\,\Gamma(z)}\right) = \Gamma(1) = 1,$$

and so

$$A = 1.$$

We have also t . = 1

$$e^b P(1) = 1$$

or, b beir P(z) and $\Gamma(z)$ are real functions,

$$b = -\log P(1) = \lim_{n=\infty} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right),$$

and so finally

$$\frac{1}{\Gamma(z)} = e^{bz} z \prod_1^\infty \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

The constant b in this formula is known as Euler's constant.

II. — The zeros of functions of finite non-integral order.

Corollary 20 constitutes a notable advance on Picard's theorem. When the function $F(z)$ is of order ρ and ρ is not an integer we can assert not merely that all the functions $F(z)-x$ without exception have an infinity of zeros, but that these are so distributed as to make the series (3, 4) divergent for $\tau<\rho$. It is possible, however, to obtain more precise results than this, and in the present section we shall consider the problem further in the light of a new idea. We shall pass over for the present the case of functions of zero order, though they properly belong to the category of the functions of this section, reserving their consideration for chapter V.

6. Proximate orders. — The results of the last section were obtained by comparing the growth of the function $\log M_1(r)$ with that of r^ρ. It is reasonable to suggest that the use of a comparison function more closely linked with $\log M_1(r)$ may lead to more precise information.

Let $F(z)$ be of finite positive order ρ. Then *there exist continuous functions $\rho(x)$, defined for $x>x_0$, differentiable in adjacent intervals and such that*

$$\overline{\lim_{x=\infty}}\, \rho(x)=\rho, \qquad \underline{\lim_{x=\infty}}\, \rho(x)\geqslant\beta,$$

$$\lim_{x=\infty}(x\rho'(x)\log x)=0$$

and

$$\overline{\lim_{r=\infty}}\frac{\log M_1(r)}{r^{\rho(r)}}=1, \quad \textit{where} \quad 0<\beta\leqslant\rho. \tag{3, 13}$$

We prove this by establishing the existence of particular functions $\rho(x)$. This could be done in an elementary manner starting from the Taylor series of an integral function. But for the sake of brevity we shall adopt a different procedure depending on the use of Blumenthal's results.

Let

$$\log M_1(r) = r^{\mu(r)}.$$

Then the upper limit of $\mu(r)$ as $r \to \infty$ is equal to the order ρ. Now we know that any interval $X_1 X$ can be divided up into a finite number of intervals $X_1, X_2, \ldots, X_p$, in each of which the function $\mu(x)$ can be expressed as a power series in $(x - X_i)^{\frac{1}{s_i}}$, the expansion being valid in the segment X_i, X_{i+1}. In such an interval the two functions

$$\mu'(x) \pm \frac{1}{x \log x \log_2 x}$$

will have a finite number of zeros, and it follows that the curve

$$\Gamma, \qquad y = \mu(x). \qquad (x > X_1)$$

will have a finite number of points of contact with the curves of the two families

$$C(\lambda) \quad y = \log_3 x + \lambda, \qquad C'(\lambda) \quad y = -\log_3 x + \lambda.$$

The angular points(¹) of the curve Γ and its points of contact with the parallel curves $C(\lambda)$ constitute a finite or infinite sequence X_1, $X'_2, \ldots, X'_n, \ldots$, and there is a similar sequence $X_1, X''_2, \ldots, X''_n \ldots$ formed with the angular points and the points of contact with the curves $C'(\lambda)$. We assume that $X_1 > e^e$ is a point beyond which Blumenthal's theorem is true, so that both sequences possess at least one member. Now consider the set of curves

$$\Gamma, \qquad y = \mu(x)$$

$$C_n \begin{cases} y_{1,n}(x) = \mu(X'_n) + \log_3 x - \log_3 X'_n & \text{for } x \leqslant X'_n \\ y_{1,n}(x) = 0 & \text{for } x > X'_n \end{cases}$$

$$C'_n \begin{cases} y_{2,n}(x) = 0 & \text{for } x < X''_n \\ y_{2,n}(x) = \mu(X''_n) + \log_3 X''_n - \log_3 x & \text{for } x \geqslant X''_n \end{cases}$$

(¹) Points at which $\mu(x)$ has unequal right and left-hand derivatives.

and the straight line $y = \beta_1 (\beta_1 < \rho)$. For every value of $x > X_1$ one of the numbers $y_{1,n}(x)$ is greater than or equal to all the rest, for, since $\mu(X'_n)$ is bounded, we can choose n so large that $y_{1,n}(x)$ will be negative, assuming of course that the sequence X'_n is infinite. If this sequence is finite the assertion is trivial. Let $Y_1(x)$ be the function of x equal to the greatest of the numbers $y_{1,n}(x)$ at each point. It is in general a discontinuous function coinciding successively with certain of the functions $y_{1,n}$, and at the points of discontinuity its right-hand value is equal to $\mu(x)$. Similarly there is a function $Y_2(x)$ equal to the greatest of the numbers $y_{2,n}$ at each point and coinciding successively with certain of the functions $y_{2,n}(x)$. Now let $\rho(x)$ be defined to be the greatest of the numbers $\mu(x)$, $Y_1(x)$, $Y_2(x)$ and β_1. $\rho(x)$ is continuous (for at their points of discontinuity the functions Y_1 and Y_2 are equal to $\mu(x)$), and coincides alternately with these four functions, unless it is ultimately identified with $\mu(x)$. If $\rho(x) = \mu(x)$ in an interval, then

$$\rho'(x) - \frac{1}{x \log x \log_2 x}$$

is continuous and does not vanish in the interval, and is therefore negative. For if it were positive $\mu(x) - \log_3 x$ would be an increasing function in the interval, which must consequently be limited. If X is the right hand limit $\mu(x) < \mu(X) + \log_3 x - \log_3 X$ and, since the point X must necessarily belong to the sequence X'_n, it would follow that $\mu(x) < Y_1(x)$ in the interval, which is contrary to our hypothesis. Similarly

$$\rho'(x) + \frac{1}{x \log x \log_2 x} > 0.$$

Therefore for all values of x

$$|\rho'(x).x \log x \log_2 x| \leqslant 1.$$

Further it is clear that $\rho(x) > \beta_1$ and $\overline{\lim}\, \rho(x) = \rho$. The conditions laid down are thus fulfilled by the function $\rho(x)$ as defined, for we have

$$\mu(x) \leqslant \rho(x);$$

and the equality holds at all events in a infinite sequence of points, and, when $\mu(x)$ is continuous, in an infinite sequence of segments.

It only remains to shew that we can find a function $\rho(x)$ tending to ρ. If the function constructed as above does not tend to ρ it will have a lower limit $\beta < \rho$ for $x = \infty$. Suppose that β' is another number lying between β and ρ. There is a sequence of points tending to infinity at each of which $\rho(x) = \mu(x) \geqslant \beta'$. Let $x_{\beta'}$ denote the first of these points. Now considering an infinite sequence of numbers β_i tending to ρ, we see that it is sufficient to replace $\rho(x)$ by the greatest of the numbers $\rho(x)$ and β_i in the interval $x_{\beta_i}, x_{\beta_{i+1}}$ to obtain a function which tends to ρ and satisfies all the above conditions.

A function $\rho(r)$ which satisfies these conditions and tends to ρ we shall call a *proximate order* L (Lindelöf) or simply an *order* L. When the order ρ is not an integer a function $\rho(r)$ such that $\underline{\lim}\, \rho(r) > p$, where p is the genus, will be called a *proximate order* B (Boutroux) or *order* B. Functions $\rho(r)$ in general will be called *general proximate orders*.

If $\rho(r)$ is an order B there exists a number α such that ultimately

$$r^{\rho(r)-p-\alpha} \quad \text{and} \quad r^{p+1-\rho(r)-\alpha}$$

are increasing functions[1]. For their differential coefficients are

$$(\rho(r) - p - \alpha + \rho'(r) r \log r)\, r^{\rho(r)-p-1-\alpha},$$
$$(p + 1 - \rho(r) - \alpha - \rho'(r) r \log r)\, r^{p-\rho(r)-\alpha}$$

and since $r \log r\, \rho'(r) \to 0$ and the limits of indetermination of $\rho(r)$ lie between p and $p+1$, α may be chosen so as to make both these differential coefficients positive.

7. The distribution of the zeros. — Let F(z) be a function of finite non-integral order ρ, $\rho(x)$ an order B and α a number such that

$$r^{\rho(r)-p-\alpha}, \qquad r^{p+1-\rho(r)-\alpha}$$

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(1) Boutroux uses this property as the definition of a function $\rho(r)$ such as we have called a proximate order B.

are increasing functions. Then, if $k > 1$, it follows from Jensen's theorem that

$$n(r)\log k < \int_0^{kr} \frac{n(x)}{x}\,dx < \log M_1(kr) + K < (kr)^{\rho(kr)} + K$$
$$< k^{p+1-\alpha} r^{\rho(r)} + K.$$

We might try to fix k so as to deduce the best possible approximation to $n(r)$; but the inequality

$$(3,\ 14) \qquad\qquad n(r) < K r^{\rho(r)} \qquad\qquad (r > r_0)$$

will be sufficient for our purpose at the moment.

Now for an indefinitely increasing sequence of values of r we have $r^{\rho(r)} = \log M_1(r)$, and so, by (3, 6),

$$r^{\rho(r)} < K r^p + K_1 \int_{r_0}^{r} \frac{n(x)\, r^{p+1}}{x^{p+1}(x+r)}\,dx.$$

Hence, for these values of r,

$$r^{\rho(r)} < K r^p + K_1 r^p \int_{r_0}^{r/\lambda} \frac{n(x)}{x^{p+1}}\,dx + K_1 r^{p+1} \int_{r\lambda}^{\infty} \frac{n(x)}{x^{p+2}}\,dx$$
$$+ K_1 n(\lambda r) \int_{r/\lambda}^{r\lambda} \frac{r^p}{x^{p+1}}\,dx$$

where $\lambda > 1$. It follows from (3, 14) and the fact that $x^{\rho(r)-p-\alpha}$ is an increasing function that the first of these integrals is less than

$$K \int_{r_0}^{r/\lambda} x^{\rho(r)-p-\alpha} x^{\alpha-1}\,dx < K_2 \frac{1}{\lambda^\alpha} r^{\rho(r)-p}.$$

Similarly, since $x^{\rho(x)-p-1+\alpha}$ is a decreasing function, the second integral is less than

$$K \int_{r\lambda}^{\infty} x^{\rho(r)-p-1+\alpha} x^{-1-\alpha}\,dx < K_3 \frac{1}{\lambda^\alpha} r^{\rho(r)-p-1}.$$

The third integral is equal to $\frac{1}{p}[\lambda^p - \lambda^{-p}]$. Therefore

$$n(r\lambda) > \frac{1}{K_3}\left[1 - \frac{K_4}{\lambda^\alpha}\right] r^{\rho(r)},$$

and finally, choosing λ such that $K_4 \lambda^{-\alpha} < \frac{1}{2}$,

$$n(r\lambda) > \frac{1}{K_6} r^{\rho(r)} = \frac{1}{K'} (r\lambda)^{\rho(r\lambda)}.$$

The following proposition is thus proved :

If $F(z)$ *is a function of non-integral order* ρ *and* $\rho(r)$ *is an order* B *of this function, the number of zeros* $n(r)$ *satisfies the inequality* (3, 14) *for all sufficiently large values of* r, *and the complementary inequality*

$$(3, 15) \qquad n(r) > \frac{1}{K'} r^{\rho(r)}$$

for a sequence of values of r *proportional to the values for which* $\log M_1(r) = r^{\rho(r)}$.

Suppose that r is a value for which inequality (3, 15) is satisfied. Plainly if h is sufficiently large the number of zeros in the circle $|z| = \frac{1}{h} r$ will be less than $\frac{1}{2} n(r)$. The number of zeros in the ring between the circles $|z| = \frac{1}{h} r$ and $|z| = r$ will then be at least equal to $Kr^{\rho(r)}$, and so equal to $K(r) r^{\rho(r)} = K_1(r) \log M_1(r)$ where $K_1(r)$ lies between two fixed positive numbers. In this form the result is independant of the idea of a proximate order.

THEOREM 21. — *If* $F(z)$ *is a function of finite non-integral order, we can find an infinite sequence of non-overlapping annular regions* D_m, $R_m \leqslant |z| \leqslant kR_m$, k *being fixed, and two positive numbers* k' *and* k'' *such that every function* $F(z) - x$ *has*

$$h \log M_1(kR_m), \qquad k' \leqslant h \leqslant k''$$

zeros in each annulus D_m, *provided* $m > m_x$.

8. The zeros of functions of regular growth. — It follows from the preceding argument that if

$$\varliminf_{r=\infty} \frac{\log M_1(r)}{r^{\rho(r)}} = D > 0,$$

where $\rho(r)$ is an order B, then the inequality (3, 15) is also valid for all sufficiently large values of r. The ratio

$$n(r)/r^{\rho(r)}$$

will then ultimately be confined between fixed positive limits. Conversely, if $\rho(r)$ is a function satisfying the conditions of growth for an order B and if $n(r) < Kr^{\rho(r)}$ for all sufficiently large values of r, it follows from (3, 6) that

$$\log M_1(r) < K'r^{\rho(r)}.$$

Similarly if for all sufficiently large values of r

$$n(r) > K\, r^{\rho(r)},$$

we have, by (3, 2),

$$\log M_1(hr) > \int_r^{hr} \frac{n(x)}{x}\, dx > n(r) \log h > \frac{1}{K'} (hr)^{\rho(hr)}.$$

If the condition for $n(r)$ is satisfied only for an infinite sequence of values of r, then the last inequality is valid only for this sequence.

To sum up, in order that we should have

$$\varliminf_{r=\infty} \frac{\log M_1(r)}{r^{\rho(r)}} > 0$$

where $\rho(r)$ is an order B, it is necessary that $n(r)$ should be equal to $h(r)\, r^{\rho(r)}$, $h(r)$ having finite positive limits of indetermination; and conversely, if this condition is fulfilled, the limits of indetermination of $\log M_1(r)/r^{\rho(r)}$ are finite and positive. In particular *the necessary and sufficient condition that a function of finite non-integral order ρ should be of very regular growth is that the ratio n/r_n^{ρ} should have finite positive limits of indetermination.*

It is also possible to prove, by a method analogous to that of § 7, that :

The necessary and sufficient condition that a function of non-integral order ρ should be of regular growth (2, 16) *is that the ratio* $\log n / \log r_n$ *should tend to the limit* ρ.

The method, which consists in applying Jensen's inequality (3, 2) to find an upper limit for $n(r)$ and then using this upper limit and the inequality (3, 6) to find a complementary lower limit to $n(r)$, is moreover a prefectly general application. But it must be assumed that ρ is not an integer.

III. — Functions of integral order and Borel's theorem.

Functions of positive integral order lie completely outside the considerations of the last section. It is still possible to find an upper limit for the number of zeros, but the argument by which we provéd theorem 21 breaks down. So long as we are in ignorance of the genus of the function we are unable to apply the inequality (3, 6); and even if the genus is known this inequality does not lead to results of the same precision. Moreover it may happen that the function has no zeros at all. It is then, in the case of an integral function of the form $e^{Q(z)}$, and we see that a function with no zeros must satisfy the condition

$$(3, 16) \qquad \frac{\log M_{t}(r)}{r^{\rho}} \rightarrow \textit{a limit.}$$

This condition is plainly also necessary if a function is to have a finite number of zeros, and we shall shew that it is satisfied in general by all functions for which the exponent of convergence of the zeros is less than the order. In fact *the only functions of integral order for which the exponent of convergence of the zeros can be less than the order are those fulfilling the condition* (3, 16).

For the sake of simplicity we restrict ourselves to integral functions. Let $f(z)$ be a function with exponent $\rho_1 < \rho$. Then

$$f(z) = e^{Q(z)} P(z) z^{\alpha},$$

$Q(z)$ being a polynomial of degree $q = \rho$ and $P(z)$ a canonical product of order $\rho_1 < \rho$. If b is the coefficient of z^q in $Q(z)$ we have, for all ε and $r > r_\iota$,

$$\log M(r) < (|b| + \varepsilon) r^q.$$

On the other hand $\mathcal{R}(Q(z)) > (|b| - \varepsilon) r^q$ in q angles of magnitude h (h depending on ε) and since, by theorem 19, the total length of the arcs of the circle $|z| = r$ on which

$$\log |P(z)| \leqslant - r^{\rho_1 + \varepsilon'}$$

does not tend to infinity with r, there are certainly points of every circle $|z| = r > r'_\iota$ at which

$$\log |f(z)| > (|b| - \varepsilon) r^q.$$

Therefore

$$\log M(r) > (|b| - \varepsilon) r^q$$

and our assertion is proved.

Borel has proved that only one of the functions $F(z) - x$ (where x may be given any value) can possibly have an exponent of convergence less than the order ρ.

9. Borel's theorem. — THEOREM 22. — *If the order of* $F(z)$ *is an integer and the condition* (3, 16) *is satisfied, then the exponent of convergence of the zeros of the functions* $F(z) - x$ *is equal to the order, save possibly for one exceptional value of* x.

Suppose that there are two such exceptional values a and b. Then

$$F(z) - a = z^{\alpha_1} \Phi_1\left(\frac{1}{z}\right) e^{Q_1(z)} P_1(z), \quad F(z) - b = z^{\alpha_2} \Phi_2\left(\frac{1}{z}\right) e^{Q_2(z)} P_2(z),$$

where Q_1 and Q_2 are polynomials of degree ρ and P_1, P_2 canonical products of order less than ρ. Subtracting the second of these two equations from the first we have

$$(\beta) \qquad \Phi_1 z^{\alpha_1} P_1 e^{Q_1} - \Phi_2 z^{\alpha_2} P_2 e^{Q_2} = b - a$$

and, multiplying by e^{-Q_2},

$$\Phi_1 z^{\alpha_1} P_1 e^{Q_1 - Q_2} = \Phi_2 z^{\alpha_2} P + (b-a) e^{-Q_2}.$$

Now, Q_2 being a polynomial of degree ρ, $\left|e^{-Q_2}\right| > e^{Kr^\rho}$. The left hand side of this equality must therefore be of order ρ and, since the factor $\Phi_1 z^{\alpha_1} P_1$ is of order $\leqslant \rho - \gamma$, it follows that $Q_1 - Q_2$ is a polynomial of degree ρ.

Differentiating the identity (β) we have

$$[\Phi_1 z^{\alpha_1} P_1 Q_1' + (\Phi_1 z^{\alpha_1} P_1)'] e^{Q_1} - [\Phi_2 z^{\alpha_2} P_2 Q_2' + (\Phi_2 z^{\alpha_2} P_2)'] e^{Q_2} = 0.$$

By theorem 13, P'_1 is of order not greater than $\rho - \gamma$. So the coefficient of e^{Q_1} is of order not greater than $\rho - \gamma$ and, unless $\Phi_1 z^{\alpha_1} P_1 e^{Q_1}$ is a constant, it is plainly not identically zero. Similarly for the coefficient of e^{Q_2} Factorising the coefficients this identity becomes

$$z^{\alpha_3} \Phi_3 P_3 e^{Q_1 + Q_3} - z^{\alpha_4} \Phi_4 e^{Q_2 + Q_4} = 0,$$

where Q_3, Q_4 are polynomials of degree not greater than $\rho - 1$ and P_3, P_4 are canonical products of order not exceeding $\rho - \gamma$. This may be written

$$e^{Q_1 - Q_2 + Q_3 - Q_4} \equiv z^{\alpha_4 - \alpha_3} \frac{\Phi_4}{\Phi_3} \frac{P_4}{P_3}$$

But, since $Q_1 - Q_2 + Q_3 - Q_4$ is a polynomial of degree ρ. the logarithm of the modulus of the left-hand expression is greater than Kr^ρ, whereas, by theorem 19, the right-hand expression, being of order not greater than $\rho - \gamma$, is less in modulus than $e^{r^{\rho-\gamma+\varepsilon}}$ on an infinite sequence of circles. We have thus arrived at a contradiction which proves the theorem.

A value of x such that the exponent of convergence of the zeros of the function $F(z) - x$ is less than the order of $F(z)$ we shall say is *exceptional* B *for* $F(z)$. Borel's theorem states that there is only one such exceptional value.

We could prove by a slight modification of our argument that if the functions $F_1(z)$ are of lower order than $F(z)$, then, of all the functions $F(z) + F_1(z)$ only one can be such that the exponent of convergence of its zeros is less than the order. The functions for which there is a value x exceptional B are therefore themselves exceptional. Such a function we shall say *is exceptional* B.

10. The order of multiplicity of the zeros. — Fatou and Montel have proved certain propositions concerning the order of multiplicity of the zeros of a function $F(z) - x$. With the aid of Borel's theorem it is possible to generalize their results.

We observe at the outset that *in order that a function* $F(z)$ *of order* ρ *should be exceptional* B *for a given value* a, *it is necessary and sufficient that there should be a function* $\Phi(z)$ *such that the product* $[F(z) - a]\,\Phi(z)$ *is of order less than* ρ. For, if such a function exists, the exponent of convergence of the zeros of $[F(z) - a]\,\Phi(z)$ is less than ρ and this is true *a fortiori* for either of the factors. Conversely, if a is a value exceptional B, there is, as we have seen, a polynomial $Q(z)$ such that the product $[F(z) - a]\,e^{-Q(z)}$ is of order less than ρ.

In a domain exterior to the circles of radius $r_n^{-h}\,(h > \rho)$ *described about the zeros* $a_n\,(r_n > 1)$ *as centres we have*

$$\left|\frac{F'(z)}{F(z)}\right| < r^K.$$

That this domain actually exists and that it contains circles $|z| = \text{const.}$ in every annulus $R_0 \leqslant R \leqslant |z| \leqslant R + 1$ was shewn in the proof of theorem 19. From the product form for $F(z)$ (1, 11) we obtain

$$\frac{F'(z)}{F(z)} = \frac{\alpha}{z} - \frac{\Phi'\left(\frac{1}{z}\right)}{z^2\,\Phi\left(\frac{1}{z}\right)} + Q'(z) + \frac{P'(z)}{P(z)}.$$

The modulus of the sum of the first three terms on the right is less than Kr^{q-1}, q being the degree of $Q(z)$. $P(z)$ is a canonical product and its logarithmic derivative is equal to the sum of the logarithmic derivatives of its factors. To see this it is sufficient to apply theorem 5 to $\log P(z)$ in a region in which it is regular. Thus

$$\frac{P'(z)}{P(z)} = \sum_1^\infty \left[\frac{1}{z-a_n} + \frac{1}{a_n} + \ldots + \frac{z^{p-1}}{a_n^{\,p}}\right] = \sum_1^\infty \frac{z^p}{(z-a_n)a_n^{\,p}}.$$

We divide this sum into two parts and, putting $N = n(kr)$, we have

$$\sum_1^N \frac{r^p}{r_n^{\,p}|r-r_n|} < r^p \sum_1^N r_n^{\,h-p} < k^{h-p} r^h N < K_1 r^{h+\rho+\epsilon}$$

for all sufficiently large values of r. Further, since $\sum \frac{1}{r_n^{\,p+1}}$ is convergent, the remainder

$$r^p \sum_{N+1}^\infty \frac{1}{r_n^{\,p}(r_n-r)} < \frac{k}{k-1}\, r^p \sum_{N+1}^\infty \frac{1}{r_n^{\,p+1}} < K_2 r^p,$$

and these two inequalities together prove the proposition.

THEOREM 23. — *If $F(z)$ is of finite order ρ there cannot be more than two values of x for which the exponent of convergence of the simple zeros of the function $F(z) - x$ is less than ρ.*

We shall say that such values are *exceptional* B *for the simple zeros*.

To prove the theorem suppose that there are three such values a, b and c. No two of the functions

$$F(z) - a, \qquad F(z) - b, \qquad F(z) - c$$

have the same zeros. A zero of $F(z) - a$ of order of multiplicity $\gamma(\gamma > 1)$ is a zero of $[F'(z)]^2$ of order $2\gamma - 2 \geqslant \gamma$. Therefore, if $P(z, a)$ is the canonical product formed with the simple zeros of $F(z) - a$ and $P(z, b)$, $P(z, c)$ are the two analogous products, the product

$$P(z, a)\, P(z, b)\, P(z, c)\, [F'(z)]^2$$

is divisible by the product

$$\left(F(z)-a\right)\left(F(z)-b\right)\left(F(z)-c\right).$$

In other words

$$\frac{P(z,a)\,P(z,b)\,P(z,c)\left(F'(z)\right)^2}{\left(F(z)-a\right)\left(F(z)-b\right)\left(F(z)-c\right)}=\theta(z)$$

where $\theta(z)$ is a function with an isolated essential singularity at infinity. The product of the functions P is by hypothesis of order less than ρ, and it follows from the preceding proposition that

$$\left|\frac{F'(z)}{F(z)-a}\right|\cdot\left|\frac{F'(z)}{F(z)-b}\right|<r^{2k}$$

on an infinite sequence of circles such that the radii of two successive members of the sequence differ by a quantity less than 1. Hence we conclude that the function $(F(z)-c)\,\theta(z)$ is of order less than ρ; and similarly for $F(z)-a$ and $F(z)-b$. The three values a, b, c are therefore all exceptional B and, since this conclusion contradicts Borel's theorem, the proposition is proved.

If a function $F(z)$ *has a value exceptional* B *for the whole aggregate of zeros, then there can be no other value exceptional only for simple zeros.* For if $F(z)-a$ is exceptional B for the whole aggregate of zeros, then so is $F'(z)$. The exponent of convergence of the whole sequence of zeros of $F(z)-x$, $x\neq a$, is equal to ρ, while the exponent of its multiple zeros is less than ρ, since $F'(z)$ is exceptional B. The sequence of simple zeros of $F(z)-x$ is therefore *normal* B — *i.e* of exponent ρ.

In this connection we can also prove a result analogous to Picard's theorem : *If the order of* $F(z)$ *is finite and not a multiple of* $\frac{1}{2}$, *there can only be one value exceptional* P *for simple zeros.* By this we mean a value x such that the function $F(z)-x$ has only a finite number of simple zeros.

For, returning to the proof of theorem 23, we see that if there are

two such values, which we may suppose to be -1 and $+1$, the function

$$\theta(z) = Q_1(z)Q_2(z)\frac{(F'(z))^2}{(F(z)-1)(F(z)+1)},$$

where Q_1, Q_2 are polynomials, has only a pole at infinity, since

$$|\theta(z)| < r^K \qquad (r > r_0).$$

Therefore

$$\frac{F'^2}{F^2-1} = R(z)$$

where $R(z)$ has a pole at infinity. If this pole is of order p we obtain, on integrating,

$$F(z) = \cos\left(\int \sqrt{R(z)}\,dz\right) = \cos\left(\sqrt{z^{p+2}}\,\Phi\left(\frac{1}{z}\right)\right),$$

where $\Phi\left(\frac{1}{z}\right)$ is regular and does not vanish at infinity. Functions with two values exceptional P are thus certainely either of integral order or of order which is an odd multiple of $\frac{1}{2}$. They also satisfy the condition (3, 16).

THEOREM 24. — *If* $F(z)$ *is of finite order* ρ *there is not more than one value* x *exceptional* B *for the joint sequence of simple and double zeros, and if such a value* x *exists the sequence of simple zeros is normal* B *for every other value.*

Suppose that for $x = a$ the sequence of simple and double zeros is exceptional B. The canonical product $P(z, a)$ formed with these simple and double zeros is then of order less than ρ. The product

$$[F'(z)]^3\,[P(z, a)]^2$$

is divisible by $(F(z)-a)^2$, for a zero of $F(z)-a$ of order $\gamma > 2$ is of order $3\gamma - 3 \geqslant 2\gamma$ in $(F'(z))^3$. If the sequence of simple zeros of

$F(z) - b$ is also exceptional B and if $P(z, b)$ is the corresponding canonical product, the quotient

$$\frac{[F'(z)]^6 \, [P(z, a)]^3 \, [P(z, b)]^3}{[F(z) - a]^3 \, [F(z) - b]^3}$$

is a function with an isolated essential singularity at infinity. A repetition of the argument of theorem 23 then leads to the impossible conclusion that both a and b are exceptional B for the whole sequence of zeros.

It is clear that exceptional values of this kind may occur whatever be the order of the function. The theorem we have just proved can be expressed in the phrase : *Borel's theorem is true when all zeros of order higher than the second are ignored.*

IV. — The distribution of the zeros of integral functions of finite order in general.

Borel's theorem does not lead to information as precise as that supplied by theorem 21. The method can be modified, however, to give a perfectly general and equally precise result in the case of an integral function $f(z)$. Instead of casting it into the form of a Weierstrassian product we put

$$f(z) = P(z)e^{\Phi(z)},$$

where $P(z)$ is a polynomial formed with the zeros lying in and on the circle $|z| = kr$ $(k > 1)$, the function $\Phi(z)$ being consequently regular in this circle. Our method, which depends upon corollary 8, theorem 13, Jensen's theorem and a result concerning the minimum modulus of a polynomial due to Boutroux, consists in applying Borel's argument to $f(z)$ expressed in this form.

11. The minimum modulus of a polynomial. — First consider a real polynomial of degree n in the real variable x,

$$g(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$$

and suppose that its zeros lie on the segment between 0 and 1. $g(x)$ is a continuous function, so that, if A is a given number, the values of x between 0 and 1 for which $|g(x)| < A$ constitute a set of not more than n intervals containing the zeros α_i. Writing

$$\Phi(x) = \log |g(x)| = \sum_1^n \log |x - \alpha_i|,$$

we have

$$\int_0^1 \Phi(x)dx = \sum_1^n \int_0^1 \log |x - \alpha_i| \, dx.$$

Now

$$\int_0^1 \log |x - \alpha_i| \, dx = -1 + \alpha_i \log \alpha_i + (1 - \alpha_i) \log (1 - \alpha_i) \geqslant -1 - \log 2,$$

and consequently

$$\int_0^1 \Phi(x)dx > -n \log (2e).$$

But we say that there are intervals in which $\Phi(x)$ is less than a given number, say $-Hn \log (2e)$. Since $\Phi(x)$ is negative in the range of integration this inequality shews that the total length of these intervals does not exceed $\frac{1}{H}(H > 1)$. Thus *if* H *is a number greater than* 1 *the polynomial satisfies the inequality*

$$|g(x)| > (2e)^{-Hn}$$

for values of x between 0 *and* 1, *save in a set of not more than n intervals of total length not exceeding* $1/H$ (¹).

Now consider a polynomial

$$P(z) = \left(1 - \frac{z}{\alpha_1}\right) \ldots\ldots \left(1 - \frac{z}{\alpha_N}\right)$$

(¹) This method of proof appears to be due to Sire.

of degree N in the complex variable z and suppose that its zeros lie in the wide sense in the circle $|z| \leqslant R$ (that is to say inside or on the circumference), and let k' be a number greater than 1. We shall shew that the preceding result can be generalized as follows. *In the annular regions*

$$R/k' \leqslant |z| \leqslant R \quad \text{and} \quad R/k'^3 \leqslant |z| \leqslant R/k'^2$$

there are circles $|z| = r$ on which

$$|P(z)| > [(k'-1)k'^{-4}(2e)^{-4k'^3}]^N . \tag{3, 17}$$

Clearly, for $|z| \geqslant Rk'^{-3}$,

$$|P(z)| > \frac{r^N}{|\alpha_1 \ldots \alpha_N|} \prod_1^N \left| 1 - \frac{|\alpha_n|}{r} \right| > k'^{-3N} \prod_1^N \left| 1 - \frac{|\alpha_n|}{r} \right| .$$

Let m be the number of zeros whose modulus does not exceed Rk'^{-4}. Then

$$\prod_1^m \left| 1 - \frac{|\alpha_n|}{r} \right| > \left(1 - \frac{1}{k'} \right)^m, \quad \prod_{m+1}^N \left| 1 - \frac{|\alpha_n|}{r} \right| > \prod_{m+1}^N \left| \frac{r}{R} - \frac{|\alpha_n|}{R} \right| .$$

Now writing

$$r = \frac{R}{k'^4} + x(1 - k'^{-4})R, \qquad |\alpha_n| = \frac{R}{k'^4} + \beta_n(1 - k'^{-4})R$$

we see that the numbers x and β_n lie between 0 and 1 when r and α_n lie between R/k'^4 and R. So

$$\prod_{m+1}^N \left| 1 - \frac{|\alpha_n|}{r} \right| > (1 - k'^{-4})^{N-m} \left| \prod_{m+1}^N (x - \beta_n) \right| .$$

The polynomial on the right is of the form considered first of all, and its modulus is therefore greater then $(2e)^{-H(N-m)}$ except in a set of intervals of length $1/H$. The corresponding values of r cover a set of

intervals of length $\frac{1}{H}R(1-k'^{-4})$ and, choosing $H = 4k'^3 > \frac{k'^4-1}{k'-1}$, this is less than $(k'-1)k'^{-4}R$, which is the breadth of the smallest of the two annuli of the theorem. Therefore in both annuli there are circles $|z| = r$, on which

$$|P(z)| > k'^{-3N}\left(\frac{k'-1}{k'}\right)^m \left(\frac{k'^4-1}{k'^4}\right)^{N-m} (2e)^{-4k'^3(N-m)},$$

and this leads to the required inequality.

12. A general theorem. — THEOREM 25. *If $f(z)$ is an integral function of finite order and $n(r, x)$ the number of zeros of the function $f(z) - x$ in the circle $|z| \leqslant r$, then to every number $k > 1$ there corresponds a positive number $H(k)$ such that, for all values of a and b $(a \neq b)$ and for all $r > r(a, b)$,*

$$(3, 18) \qquad \int_0^r \frac{n(x, a) + n(x, b)}{x}\,dx > H(k)\log M\left(\frac{r}{k}\right) \qquad (\alpha > 0),$$

where $M(r)$ is the maximum modulus of $f(z)$ on the circle of radius r.

For the sake of simplicity(1) we suppose that $f(0) \neq a$, $f(0) \neq b$ and $f'(0) \neq 0$ and we write

$$f(z) - a = f_1(z) = c_0 + c_1 z + \dots$$
$$f(z) - b = f_2(z) = c_0' + c_1 z + \dots,$$

where c_0, c'_0, and c_1 are not zero.

Let us assume that the theorem is false. Then we can find a large value of r, say R, for which the inequality (3, 18) is invalid. If we put

$$R_1 = \frac{R}{k'^7}, \quad R_2 = \frac{R}{k'^6}, \quad \dots, \quad R_7 = \frac{R}{k'}, \quad R_8 = R \qquad (k'^7 = k)$$

this implies

$$\int_0^{R_8} \frac{n(x, a)}{x}\,dx \leqslant H(k)\log M(R_1)$$

(1) The particular case $c_0 = c_1 = \dots = c_{p-1} = 0$ can be dealt with in a similar manner.

since $n(x, b)$ is positive, and hence

$$n(R_7, a) \leqslant \frac{H}{\log k'} \log M(R_1), \qquad H = H(k).$$

The function f_1 can be expressed in the form

$$f_1(z) = c_0 P_1(z) e^{\Phi_1(z)} \tag{d}$$

where $P_1(z)$ is a polynomial with the same zeros inside the circle $|z| \leqslant R_7$ as $f_1(z)$ and equal to 1 at the origin. $\Phi_1(z)$ is regular in this circle and zero at the origin. If $r_1, r_2, \ldots, r_N$ are the moduli of these zeros of $f_1(z)$, we have

$$\log \frac{R_7^N}{r_1 r_2 \ldots r_N} = \int_0^{R_7} \frac{n(x, a)}{x} dx \leqslant H \log M(R_1),$$

where

$$N = n(R_7, a).$$

Consequently(¹), for $|z| = r \leqslant R_7$,

$$\text{(e)} \quad |P_1(z)| \leqslant M_1(R_7) \leqslant \frac{R_7^N}{r_1 r_2 \ldots r_N} \prod_1^N \left(1 + \frac{r_n}{R_7}\right) < M(R_1)^H \prod_1^N \left(1 + \frac{r_n}{R_7}\right)$$

$$< 2^N M(R_1)^H < M(R_1)^{H\left(1 + \frac{\log 2}{\log k'}\right)}.$$

On the other hand we know, by the result of § 11, that there is at least one circle in the annulus $R_0 \leqslant |z| \leqslant R_7$ on which

$$|P_1(z)| > e^{-NH'} > M(R_1)^{-\frac{HH'}{\log k'}} \tag{i}$$

$$\left(H' = 4 \log k' + 4k'' \log (2e) - \log (k' - 1)\right).$$

Denoting by $A_1(r)$ the maximum real part of $\Phi_1(z)$, ($|z| = r$), we have, for sufficiently large values of R, say $R > R_a$, as a consequence of (d) and (e), the inequality,

$$e^{A_1(R_1)} > \frac{1}{|c_0|} (M(R_1) - |a|) M(R_1)^{-H\left(1 + \frac{\log 2}{\log k'}\right)} > M(R_1)^{1 - H\left(2 + \frac{\log 2}{\log k'}\right)}$$

(¹) In this paragraph M_1 denotes the maximum modulus of P_1.

Similarly it follows from (d) and (i) that

$$\text{(l)} \qquad A_1(R_6) < \left(2 + \frac{HH'}{\log k'}\right) \log M(R_7).$$

Now $f(z)$ is by hypothesis of finite order, so that, supposing k' fixed and R sufficiently great, we shall have

$$\text{(m)} \qquad A_1(R_6) < KR_7^{\rho + \frac{1}{2}} < R_1^{\rho+1}$$

and, by corollary 8,

$$\text{(n)} \qquad |\Phi_1(z)| < R_1^{\rho+2} \qquad \text{for} \qquad |z| = r \leqslant \sqrt{R_5 R_6}.$$

Applying the theorem on the derivative (theorem 13) to the functions $\Phi_1(z)$ and $P_1(z)$ we find, for $r \leqslant R_5$

$$\text{(o)} \qquad \begin{cases} |\Phi_1'(z)| < R_1^{\rho+3}, \\ |P_1'(z)| < M(R_1)^{H\left(1+\frac{\log 2}{\log k'}\right)}. \end{cases}$$

Precisely similar results can be obtained for the functions $\Phi_2(z)$ and $P_2(z)$, when

$$f_2(z) = c_0' P_2(z) e^{\Phi_2(z)}.$$

From this point our argument proceeds along the lines laid down by Borel. Thus

$$c_0 P_1(z) e^{\Phi_1(z)} - c_0' P_2(z) e^{\Phi_2(z)} = b - a = c_0 - c_0'$$

and, differentiating,

$$c_0(P_1' + \Phi_1' P_1) e^{\Phi_1} - c_0'(P_2' + \Phi_2' P_2) e^{\Phi_2} = 0.$$

The coefficient of e^{Φ_2} does not vanish identically, since

$$c_0'(P_2' + \Phi_2' P_2) = c_1 + \ldots .$$

Solving this equation in e^{Φ_2} and substituting in the preceding one we obtain

$$\text{(p)} \qquad \Phi_3 e^{\Phi_1} = c_0'(P_2' + \Phi_2' P_2)$$

where

$$\Phi_3 = \frac{c_0 c_0'}{c_0 - c_0'} [P_1(P_2' + \Phi_2' P_2) - P_2(P_1' + \Phi_1' P_1)].$$

This function $\Phi_3(z)$ is regular in the circle $r \leqslant R_7$ and $\Phi_3(o) = c_1$. Further the inequalities (e) and (o) shew that

$$\log |\Phi_3(z)| < 2H\left(2 + \frac{\log 2}{\log k'}\right) \log M(R_1)$$

for $r \leqslant R_6$.

$\Phi_3(z)$ can be expressed as a product

$$\Phi_3(z) = c_1 P_3(z) e^{\Phi_4(z)}$$

where $P_3(z)$ is a polynomial with the same zeros in the circle $|z| < R_4$ as $\Phi_3(z)$ and such that $P_3(o) = 1$. The number N' of these zeros satisfies Jensen's inequality and we have

$$N' < HH'' \log M(R_1) \qquad \left(H'' = \frac{2}{\log k'}\left(2 + \frac{\log 2}{\log k'}\right)\right).$$

It follows from this inequality and the theorem concerning the modulus of the polynomial that there is at least one circle between $|z| = R_3$ and $|z| = R_4$, on which

$$\text{(s)} \qquad \left|\frac{1}{P_3(z)}\right| < M(R_1)^{HH'H''}.$$

Hence on this circle

$$\left|e^{\Phi_4(z)}\right| = \left|\frac{1}{c_1} \frac{\Phi_3(z)}{P_3(z)}\right| < M(R_1)^{HH'''}$$

$$\left(H''' = 5 + \frac{\log 4}{\log k'} + H'H''\right)$$

or

$$A_4(R_3) < H H''' \log M(R_1),$$

where $A_4(r)$ is the maximum real part of $\Phi_4(z)$ on $|z| = r$.

Therefore, in virtue of corollary 8,

$$|\Phi_4(z)| < \frac{4}{k'-1} H H''' \log M(R_1) \qquad \text{for} \qquad |z| \leqslant R_2.$$

Now equation (p) may be written

$$e^{\Phi_4(z)} = \frac{c_0'(P_2' + \Phi_2'P_2)}{\Phi_3(z)} = \frac{c_0'}{c_f} e^{-\Phi_4(z)} \frac{P_2' + \Phi_2'P_2}{P_1}$$

and $1/|P_3|$ satisfies the inequality (s) at least once in the annulus R_1, R_2 also. Hence, since the upper limit of $|P'_2 + \Phi'_2 P_2|$ is the same as that of $|\Phi_1(z)|$ and $A_1(r)$ is an increasing function, we see, recalling the inequality for $|\Phi_4|$, that

$$A_1(R_1) \leq H \log M(R_1) \left(\frac{4}{k'-1} H''' + 5 + \frac{\log 4}{\log k'} + H'H''\right).$$

But, if $H(k)$ is chosen so that

$$H(k)\left[7 + \frac{\log 8}{\log k'} + \frac{4}{k'-1}\left(5 + \frac{\log 4}{\log k'}\right) + \frac{k'+3}{k'-1}\,\frac{H'}{\log k'}\left(4 + \frac{\log 4}{\log k'}\right)\right] < 1$$

this is in contradiction with the inequality (j). Thus the theorem is proved.

13. Deductions from the general theorem. — Let $\rho(r)$ be a general proximate order of the function $f(z)$. Then, for an indefinitely increasing sequence of values of r,

$$\log M\left(\frac{r}{k}\right) > H_1(k)\, r^{\rho(r)},$$

There are now two possible alternatives — either for all sufficently great values of $r (r > r_0)$ in this sequence and for every a,

$$\int_0^r \frac{n(x, a)}{x}\, dx > \frac{H(k)H_1(k)}{2} r^{\rho(r)},$$

or there is at least one value a and a sub-sequence of these values of r for which this inequality is invalid, though it will be true in the particular subsequence for every other value b. Now this in conjunction with the general inequality (3, 14)

$$n(r, b) < \mathrm{K}\, r^{\rho(r)} \qquad (r > r_b),$$

where K is independant of b, gives

$$\mathrm{K}_b + \mathrm{K}\int_{r_b}^{r/\lambda} x^{\rho(r)-1}\,dx + n(r, b)\log\lambda > \frac{\mathrm{H}(k)\,\mathrm{H}_1(k)}{2}\,r^{\rho(r)} \qquad (\lambda > 1)$$

for an infinite sequence of values of r. But, since $\rho(r)$ is a proximate order,

$$\int_{r_b}^{r/\lambda} x^{\rho(r)-1}\,dx < r^{\rho(r)-\alpha}\int_{r_0}^{r/\lambda} x^{\alpha-1}\,dx < \frac{1}{\alpha\lambda^{\alpha}}\,r^{\rho(r)},$$

and consequently

$$n(r, b) > \frac{1}{\log\lambda}\left(\frac{\mathrm{H}(k)\,\mathrm{H}_1(k)}{2} - \frac{\mathrm{K}}{\alpha\lambda^{\alpha}}\right) r^{\rho(r)} - \frac{\mathrm{K}_b}{\log\lambda}.$$

Clearly λ may be chosen so that the coefficient of $r^{\rho(r)}$ shall be positive. Theorem 21 and the more precise result concerning functions of regular growth can thus be extended to the case of functions of integral order.

THEOREM 26. — *If $f(z)$ is of integral order we can find an infinite sequence of non-overlapping annular regions D_m, $\mathrm{R}_m \leqslant |z| \leqslant k\mathrm{R}_m$, k being fixed, and two positive numbers k' and k'' such that in each annulus D_m of rank $m > m_\varepsilon$, every function $f(z) - x$, save possibly for a single exceptional value of x, has*

$$h\log \mathrm{M}(k\mathrm{R}_m), \qquad k' \leqslant h \leqslant k'',$$

zeros.

Suppose now that the function $f(z)$ is of regular growth with respect to the proximate order. That is to say

$$\varliminf_{r=\infty} \frac{\log \mathrm{M}(r)}{r^{\rho(r)}} > 0.$$

In this case it follows from the preceding argument that *if there is a number a for which the ratio $n(r, a)/r^{\rho(r)}$ tends to zero, then for every other value x the ratio $n(r, x)/r^{\rho(r)}$ is confined between fixed positive limits*.

It is clear, however, that the importance of theorem 25, from which we have deduced the two last results, is not restricted to the province of functions of integral order.

Let $n(r, a, b)$ be the number of zeros of a product of the form $(f(z) - a)(f(z) - b)$ in the circle of radius r. Then we have

$$\int_0^r \frac{n(x, a, b)}{x} dx > \mathrm{H}(k) \log \mathrm{M}\left(\frac{r}{k}\right)$$

and, a fortiori,

$$n(r, a, b) > \mathrm{K} \frac{\log \mathrm{M}\left(\frac{r}{k}\right)}{\log r}.$$

This result is independent of the nature of the order so long at it is finite. An analogous theorem for functions of infinite order will be proved in the next chapter.

14. The genus of functions of integral order. — The determination of the genus of a function of integral order is attended by peculiar difficulties. Certain results bearing on this question have however been obtained by Lindelöf and they can be proved with the help of the result of the last paragraph.

Consider the function

$$f(z) = \prod_2^\infty \left(1 + \frac{z}{n(\log n)^\alpha}\right) \qquad (\alpha > 1)$$

of genus 0.

Then $n(x) \backsim \dfrac{x}{(\log x)^\alpha}$ and, integrating by parts,

$$\log \mathrm{M}(r) = \sum \log\left(1 + \frac{r}{r_n}\right) = \int_2^\infty \log\left(1 + \frac{r}{r_n}\right) dn$$

$$= \lim_{n=\infty}\left[n \log\left(1 + \frac{r}{r_n}\right) + \int_0^{r_n} \frac{r\,n(x)}{x(x+r)} dx\right] = \int_0^\infty \frac{r\,n(x)}{x(x+r)} dx.$$

Hence, substituting for $n(x)$,

$$\log M(r) \backsim \int_{r_0}^{r} \frac{r\,dx}{(x+r)(\log x)^{\alpha}}\,dx.$$

We treat this integral in two parts. If k is fixed

$$r\int_{r_0}^{kr} \frac{dx}{(x+r)(\log x)^{\alpha}} = h_1 \int_{r_0}^{kr} d\left(\frac{x}{(\log x)^{\alpha}}\right) = h_2 \frac{kr}{(\log r)^{\alpha}},$$

where h_2 is confined between two fixed limits near to $\dfrac{1}{1+k}$ and 1 respectively. For the remainder of the integral we have

$$r\int_{kr}^{\infty} \frac{dx}{(x+r)(\log x)^{\alpha}} = rh_3 \int_{kr}^{\infty} \frac{dx}{x(\log x)^{\alpha}} = h_3 \frac{r(\log r)^{1-\alpha}}{\alpha-1},$$

where h_3 lies between $\dfrac{k}{k+1}$ and 1. Since k may be chosen as large as we please it follows that

$$\log M(r) \backsim \frac{1}{\alpha-1} r(\log r)^{1-\alpha}.$$

Now let

$$r^{\rho(r)} = r(\log r)^{1-\alpha}.$$

We know that $n(r, 0) \backsim r(\log r)^{-\alpha}$, and so $n(r, 0)/r^{\rho(r)}$ tends to zero. Consequently, for all $a \neq 0$, the ratio $n(r, a)/r(\log r)^{1-\alpha}$ has finite positive limits of indetermination.

Therefore if $1 < \alpha < 2$ *and* $a \neq 0$ *the series* $\sum \frac{1}{r_n}$ *of the reciprocals of the moduli of the zeros of* $f(z) - a$ *is divergent and the function* $f(z) - a$ *is of genus* 1, *while* $f(z)$ *is of genus* 0[1].

It can be shewn that the derivative is also of genus 0. Our proof of this depends upon two propositions of some general interest quite apart from the particular application.

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[1] The genus, like the exponent of convergence, is really determined by the moduli of the coefficients. All the functions $f(z) - x$, save possibly one are of the same genus. [See G. Valiron, *Sur les fonctions entières d'ordre entier*. Comptes Rendus, April 1922, and Appendix B.]

THEOREM 27. — *If $f(z)$ is of finite order and k any number greater than 1, we can find a corresponding number $H(k)$ such that the annulus $R \leqslant |z| \leqslant kR$ contains at least one circle $|z| = r$ on which the inequality*

$$|f(z)| > [M(kr)]^{-H}$$

is satisfied.

We can clearly suppose $f(0) = 1$. Then, as in § 12,

$$f(z) = P(z)e^{\Phi(z)},$$

where $P(z)$ is the polynomial formed with the zeros of $f(z)$ which lie in the circle of radius $Rk^{-1/4}$, and such that $P(0) = 1$. The number of these zeros does not exceed $\dfrac{4}{\log k}\log M(R)$ and it therefore follows from the theorem on the minimum modulus of a polynomial that the two annuli Rk^{-1}, $Rk^{-3/4}$ and $Rk^{-1/2}$, $Rk^{-1/4}$ each contains a circle on which

$$|P(z)| > M(R)^{-H'},$$

where H' is a positive number depending on k. On the circle in the second annulus

$$\left|e^{\Phi(z)}\right| < M(R)^{1+H'},$$

and so, if $A(r)$ and $B(r)$ denote respectively the maximum and minimum real part of $\Phi(z)$ on $|z| = r$,

$$A(Rk^{-1/2}) < (1 + H')\log M(R)$$

and

$$B(Rk^{-3/4}) < H''\log M(R).$$

Therefore on the circle in the annulus Rk^{-1}, $Rk^{-3/4}$ we have

$$|f(z)| > M(R)^{-(H'+H'')},$$

which proves the theorem.

This proposition completes the earlier result concerning the minimum modulus of a canonical product (theorem 19).

Now *suppose that the function* F(z), *of integral order* ρ, *is such that* $\log M_1(r)/r^\rho$ *tends to zero and the series* $\sum \frac{1}{r_n^\rho}$ *is convergent.* F(z) can be expressed in the form

$$F(z) = \Phi\left(\frac{1}{z}\right) P(z) e^{Q(z)},$$

and

$$|e^{Q(z)}| = |F(z)| \left| \frac{1}{\Phi\left(\frac{1}{z}\right)} \right| \left| \frac{1}{P(z)} \right|.$$

P(z) is a canonical product and as we saw in the proof of theorem 18 (equation (3, 8)),

$$\lim_{r=\infty} \frac{\log |P(z)|}{r^\rho} = 0.$$

The same is therefore true of $1/|P(z)|$ on an infinite sequence of circles and so, in virtue of our hypothesis concerning $\log |F(z)|$, we have on these circles

$$\lim \frac{\max \Re(Q(z))}{r^\rho} = 0.$$

The polynomial Q(z) is therefore of degree less than or equal to $\rho - 1$. *Thus, with the conditions stated,* F(z) *is of genus* $\rho - 1$.

We can now prove the theorem we had in view.

Theorem 28. — *If* $f(z)$ *is a real integral function of genus* 0 *or* 1 *with real zeros, its derivative* $f'(z)$ *is of the same genus as* $f(z)$ *and its zeros are also real and are separated from one other by zeros of* $f(z)$.

Such a function is of the form

$$f(z) = e^{bz} z^\alpha \prod_1^\infty \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n}},$$

where $\alpha > 0$ and the numbers b and a_n are real. Differentiating logarithmically,

$$\frac{f'(z)}{f(z)} = b + \frac{\alpha}{z} + \sum_1^{\infty}\left(\frac{1}{z-a_n} + \frac{1}{a_n}\right)$$

and writing $z = x + iy$, we see that the imaginary part of $f'(z)/f(z)$ is equal to

$$-iy\left[\frac{\alpha}{x^2+y^2} + \sum_1^{\infty}\frac{1}{(x-a_n)^2+y^2}\right]$$

Since the factor in the bracket is positive it follows that $f'(z)$ can only vanish when $y = 0$. All the zeros of $f'(z)$ are therefore real. With $y = 0$, we have

$$\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = -\frac{\alpha}{x^2} - \sum_1^{\infty}\frac{1}{(x-a_n)^2}.$$

The right-hand side of this equation is negative, so that $f'(x)/f(x)$ is a decreasing function and cannot vanish more than once between a_n and a_{n+1}. Consecutive zeros of $f'(z)$ are thus separated by those of $f(z)$ and it follows that the canonical product formed with the zeros of $f'(z)$ is of the same genus as that formed with the zeros of $f(z)$. It is known that if the order of $f(z)$ is not an integer then the genus of $f(z)$ is equal to the genus of $f'(z)$ (theorem 13). To complete the proof of the theorem suppose that the order of $f(z)$ is 1 and that the genus is 0. Then

$$\lim \frac{\log |f(x)|}{r} = 0$$

and, by theorem 13, the same is true of $f'(z)$, which is therefore of genus 0, since the genus of its canonical product is also equal to 0. Similarly if $f'(z)$ is of genus 0, then so is $f(z)$.

On applying this theorem of Laguerre to the function $f(z)$ considered at the opening of this paragraph we see that its derivative is of genus 0 : — *If* $1 < \alpha < 2$ *the functions* $f(z) - a$, $a \neq 0$, *are of genus* 1, *and their derivative* $f'(z)$ *is of genus* 0.

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CHAPTER IV

On the values assumed by an integral function in the neighbourhood of points of maximum modulus.

It has been shewn by Wiman that the relations between the functions $M(r)$, $M'(r)$, $A(r)$ and $m(r)$ established in chapters II and III can in general be replaced by relations considerably more precise. Pursuing his method we are led to certain deep results concerning the behaviour of the function $f(z)$ in the neighbourhood of points z_0 at which $|f(z_0)| = M(r)$ $(|z_0| = r)$ and the growth of $M(r)$. These results enable us to give a direct proof of Picard's theorem and to extend Borel's theorems to fonctions of infinite order. In this chapter we shall also discuss the solutions of an extensive class of differential equations and we shall find that they are necessarily of regular growth in the neighbourhood of an isolated essential singularity.

I. — Fundamental inequalities and theorems of Borel.

1. Ordinary intervals. — We have recourse once more to the Newton's polygon defined by means of the coefficients in the Taylor series of the function $f(z)$ (§ II, 4) and we compare it with the polygon corresponding to a known function of simple growth. This amounts to comparing the coefficients of the two functions.

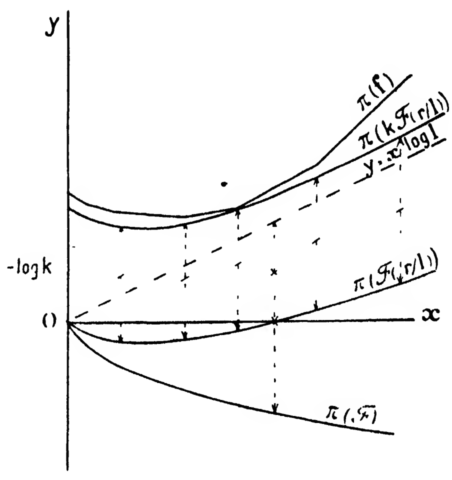
Let $\mathcal{F}(u)$ be a power series in the positive real variable u with positive, increasing, unbounded coefficients and radius of convergence unity.

$$(4, 1) \qquad \mathcal{F}(u) = \sum_{1}^{\infty} e^{H(n)} u^n,$$

where the numbers $H(n)$ tend to infinity and $H(n)/n$ tends to zero.

We suppose further that the function $H(x)$ has a decreasing derivative, which plainly tends to zero. It is clear that the points B_n of coordinates $x_n = n$, $y_n = -H(n)$ are the vertices of a polygon concave in the positive direction of y. Let this polygon be denoted by $\pi(\mathfrak{F})$. For every value of u less than 1 the function $\mathfrak{F}(u)$ has a maximum term which can be determined, as explained in chapter II, by finding the tangent to $\pi(\mathfrak{F})$ of slope $\log u$.

Now let l be any positive number and r any number less than l. To the series $\mathfrak{F}(r/l)$, regarded as a function of r, there corresponds a polygon $\pi[\mathfrak{F}(r/l)]$ derived from $\pi(\mathfrak{F})$ by adding $n \log l$ to the ordinate of each B_n. The slope of the sides of $\pi[\mathfrak{F}(r/l)]$ is an increasing function tending to $\log l$. Since the slope of the sides of $\pi(f)$ tends to infinity the polygon $\pi[\mathfrak{F}(r/l)]$ ultimately lies below $\pi(f)$. Hence, as is evident from the figure, a translation of $\pi[\mathfrak{F}(r/l)]$ parallel



to Oy can be effected so that in its new position no vertex of $\pi[\mathfrak{F}(r/l)]$ lies above the corresponding vertex of $\pi(f)$ whilst these two polygons have at least one vertex in common. If this translation is denoted by $-\log k(l)$ the polygon in its new position corresponds to the coefficients of $k(l)\mathfrak{F}(r/l)$ regarded as a function of r. It will therefore be denoted by $\pi[k\mathfrak{F}(r/l)]$.

The polygons $\pi(f)$ and $\pi[k\mathfrak{F}(r/l)]$ have one or more common vertices. Let $(n(l, \mathfrak{F})$ be the greatest of the abscissae of such vertices.

Now at a common vertex every tangent to $\pi[k\mathfrak{F}(r/l)]$ is also a tangent to $\pi(f)$. In particular the line of slope $\log r(l)$ given by the equation

$$(4,2) \qquad \log r(l) = \log l - \mathrm{H}'\big(n(l, \mathfrak{F})\big),$$

which is a tangent to $\pi[k\mathfrak{F}(r/l)]$ at the point of abscissa $n(l, \mathfrak{F})$, is also a tangent to $\pi(f)$. For this value $r(l)$ of r the maximum terms of the two functions $f(z)$ and $k\mathfrak{F}(r/l)$ are equal and of the same rank, whilst the second function dominates the first.

If l increases continuously the function $n(l, \mathfrak{F})$ cannot decrease. For the slope of a side of $\pi[k\mathfrak{F}(r/l)]$ of fixed rank increases indefinitely with l. Thus $n(l, \mathfrak{F})$ cannot be bounded. Similarly $k(l)$ is an unbounded increasing function of l. $\mathrm{H}'(n(l, \mathfrak{F}))$ is thus a decreasing discontinuous function of l which tends to zero as l tends to infinity. It follows from (4, 2) that $r(l)$ is an unbounded increasing function of l and that its only discontinuities are those of H'. These occur where l is such that $n(l, \mathfrak{F})$ is discontinous and thus correspond to those values of l for which the polygons $\pi(k\mathfrak{F}(r/l))$ and $\pi(f)$ have several common vertices. The total number of such discontinuities between 0 and $r(l)$ is not greater than $n(l, \mathfrak{F})$ and their sum does not exceed the total variation of $\mathrm{H}'(x)$; and this is finite. Hence we have the following theorem :

Given any integral function $f(z)$ and a power series $\mathfrak{F}(u)$ of the form (4, 1), we can in general find two numbers k and l corresponding to a value of r, such that, for this value of r, the maximum terms of the two functions $f(z)$ and $k\mathfrak{F}(r/l)$ are equal and of the same rank and the first function is dominated by the second. The values of r in the segment (0, R) *for which this property does not hold good constitute a set of not more than* N(R) *intervals, such that, for all values of* R, *the total variation of $\log r$ in these intervals is finite.*

Here N(R) is the rank of the maximum term of $f(z)$ for $|z| = \mathrm{R}$.

Those values of r for which the property holds good we shall call *ordinary values*, it being of course understood that they are *ordinary* with respect to a given comparison function $\mathfrak{F}(u)$. Values which are not ordinary we call *exceptional.*

In connection with ordinary and exceptional values the following observation is of interest. Suppose that for a certain value l the polygons $\pi(f)$ and $\pi[k\mathfrak{F}(r/l]$ have more than one common vertex.

Then, if the slope of the side of $\pi[k\mathfrak{F}(r/l)]$ immediately to the right of the first common vertex is equal to $\log \mathrm{R}$ and that of the side immediately to the left of the second common vertex is equal to $\log \mathrm{R}'$, the interval R, R′ is an exceptional interval. For if l is increased the change in the ordinate of the second of these vertices will be greater than the change in the ordinate of the first, and it is therefore only possible for the second to be once more a common vertex; and the slope of the side immediately to the left will then be greater than $\log \mathrm{R}'$. Similarly if l is diminished it is only possible for the first of these vertices to become once more a common vertex, and the slope of the side immediately to the right will now be less than $\log \mathrm{R}$. Repeating this argument for the second and third common vertices and for all subsequent pairs we find an exceptional interval associated with each such pair. If now we replace the polygon $\pi(f)$ by $\pi[k\mathfrak{F}(r/l)]$ in the interval between the two extreme common vertices and repeat this process in every exceptional interval of values of r, we obtain a new polygon corresponding to a function $\mathrm{U}(r, \mathfrak{F})$ with positive coefficients. $\mathrm{U}(r, \mathfrak{F})$ has the same maximum term as $f(z)$ in ordinary segments and has no exceptional intervals. $\mathrm{U}(r, \mathfrak{F})$ will be called the *dominant* $\mathfrak{F}$ *of* $f(z)$, since plainly

$$\mathrm{U}(r, \mathfrak{F}) \gg f(z).$$

Henceforth we shall take as comparison function

$$\mathfrak{F}_\alpha(u) = \sum_1^\infty e^{n^\alpha} u^n \qquad (0 < \alpha < 1) \tag{4, 3}$$

which fulfils all the conditions imposed. The corresponding dominant function will be *the dominant of index* α, $\mathrm{U}_\alpha(r)$. The corresponding ordinary values we shall call *ordinary values of index* α.

We can now shew that if *q is a positive number and r a sufficiently large ordinary value of index α, then, provided* $0 < v < \mathrm{KN}^{\alpha/2-1}$

$$\sum_{p=1}^\infty \mathrm{C}_{\mathrm{N}+p}\, p^q (1+v)^p r^{\mathrm{N}+p} < \mathrm{A}_q m(r) \mathrm{N}^{(q+1)\left(1-\frac{\alpha}{2}\right)}, \tag{4, 4}$$

$$\sum_{p=1}^{\mathrm{N}} \mathrm{C}_{\mathrm{N}-p}\, p^q (1+v)^p r^{\mathrm{N}-p} < \mathrm{A}_q m(r) \mathrm{N}^{(q+1)\left(1-\frac{\alpha}{2}\right)}, \tag{4, 5}$$

where $m(r)$ is the maximum term of $f(z)$, $\mathrm{N} = \mathrm{N}(r)$ the rank of $m(r)$ and Λ_q a constant depending on q.

Consider the first inequality. If r is an ordinary value $f(z)$ is dominated by $k\mathfrak{F}_\alpha(r/l)$ and these two functions have the same maximum term. The function on the left-hand side is thus less than

$$k\sum_{1}^{\infty} p^q(1+v)^p e^{(\mathrm{N}+p)^\alpha}\left(\frac{r}{l}\right)^{\mathrm{N}+p} = m(r)\sum_{1}^{\infty} p^q(1+v)^p e^{(\mathrm{N}+p)^\alpha-\mathrm{N}^\alpha}u^p$$

$$= m(r)\sum_{1}^{\infty} w_p.$$

Here N is the rank of the maximum term and we can therefore determine u as a function of N. In fact we have

$$\alpha(\mathrm{N}+1)^{\alpha-1} < -\log u < \alpha(\mathrm{N}-1)^{\alpha-1}$$

so that

$$\log u < -\alpha(\mathrm{N}+1)^{\alpha-1}.$$

Now let N_1 be the greatest integer less than $\mathrm{B}_q\mathrm{N}^{1-\alpha/2}$, where B_q is a number which we shall fix later. We have

$$\log w_{2\mathrm{N}_1+i} - \log w_{\mathrm{N}_1+i}$$
$$= q\log\frac{2\mathrm{N}_1+i}{\mathrm{N}_1+i} + \mathrm{N}_1\log(1+v) + \mathrm{N}_1\log u + (2\mathrm{N}_1+\mathrm{N}+i)^\alpha - (\mathrm{N}_1+\mathrm{N}+i)^\alpha.$$

The first term on the right is clearly less than $q\log 2$ and the second is less than $\mathrm{N}_1 v$ and therefore, in virtue of the condition imposed on v, less than $\mathrm{B}_q\mathrm{K}$. The sum of the last three terms is shewn by the mean value theorem to be less than

$$\begin{aligned}
&(2\mathrm{N}_1+\mathrm{N}+i)^\alpha - (\mathrm{N}_1+\mathrm{N}+i)^\alpha - \alpha(\mathrm{N}+1)^{\alpha-1}\mathrm{N}_1 \\
&\quad = \alpha\mathrm{N}_1[(\mathrm{N}+\mathrm{N}_1(1+\theta)+i)^{\alpha-1} - (\mathrm{N}+1)^{\alpha-1}] \qquad (0<\theta<1) \\
&\qquad < \alpha\mathrm{N}_1[(\mathrm{N}+\mathrm{N}_1)^{\alpha-1} - (\mathrm{N}+1)^{\alpha-1}] \\
&\quad = \alpha(\alpha-1)\mathrm{N}_1(\mathrm{N}_1-1)[\mathrm{N}+1+\theta'(\mathrm{N}_1-1)]^{\alpha-2} \qquad (0<\theta'<1) \\
&\qquad < -\alpha(1-\alpha)(\mathrm{N}_1-1)^2(\mathrm{N}+\mathrm{N}_1)^{\alpha-2}.
\end{aligned}$$

Thus the logarithm of the ratio of the terms of rank $2N_1 + i$ and $N_1 + i$ satisfies the inequality

$$\log w_{2N_1+i} - \log w_{N_1+i} < q \log 2 + B_q K - \alpha(1-\alpha) B_q^2 \lambda,$$

where λ tends to 1 as N tends to ∞. It is clear that B_q may be chosen so as to make the right-hand side of this inequality less than -1. If S is the sum of the series Σw_p and S_p is the sum of the first p terms of this series we have

$$S - S_{2N_1} < \frac{1}{e}(S - S_{N_1}),$$

and so

$$\frac{e-1}{e} S < S_{2N_1} - \frac{1}{e} S_{N_1} < S_{2N_1}.$$

Therefore, since the numbers $e^{(N+p)^2 - N^2} u^p$ are less than 1,

$$S < \frac{2e}{e-1} N_1^{q+1} (1+v)^{2N_1} < \frac{2e}{e-1} N_1^{q+1} e^{2N_1 v} < A_q N^{(q+1)\left(1-\frac{\alpha}{2}\right)},$$

and the inequality (4, 4) is established. The second result can be proved by a precisely similar argument.

2. Some fundamental inequalities. — Let $f(z)$ be an integral function, z and z_0 any two values of the variable and n a positive integer. Then we may write

$$f(z) = \sum_{p=-n}^{\infty} c_{n+p} z_0^{n+p} \left(\frac{z}{z_0}\right)^{n+p} = \left(\frac{z}{z_0}\right)^n \sum_{-n}^{\infty} c_{n+p} z_0^{n+p} \left(\frac{z}{z_0}\right)^p$$

and, putting

$$\left(\frac{z}{z_0}\right)^p = \left(1 + \frac{z-z_0}{z_0}\right)^p = 1 + p\frac{z-z_0}{z_0} + \left(\frac{z-z_0}{z_0}\right)^2 \chi_p\left(\frac{z-z_0}{z_0}\right),$$

we have

$$(4, 6) \quad f(z) = \left(\frac{z}{z_0}\right)^n \left[f(z_0) + \frac{z-z_0}{z_0} g(z_0) + \left(\frac{z-z_0}{z_0}\right)^2 \chi(z, z_0)\right],$$

where

$$(4, 7) \qquad g(z) = zf'(z) - nf(z),$$

$$|\chi(z, z_0)| < \sum_{-n}^{\infty} C_{n+p}|z_0|^{n+p}\left|\chi_p\left(\frac{z-z_0}{z_0}\right)\right|.$$

Now applying the binomial theorem and the remainder form of Taylor's theorem we see that, *if p is a positive integer*

$$|u^2\chi_p(u)| = |(1+u)^p - 1 - pu| \leqslant (1+|u|)^p - 1 - p|u|$$
$$= \frac{p(p-1)}{2}(1+\theta|u|)^{p-2}|u|^2 < \frac{1}{2}p^2(1+|u|)^p|u|^2,$$

whilst, *if p is a negative integer and* $|u| < 1$,

$$|u^2\chi_p(u)| \leqslant \left|(1-|u|)^p - 1 + p|u|\right| = \left|\frac{p(p-1)}{2}(1-\theta|u|)^{p-2}|u|^2\right|$$
$$< \frac{1}{2}p^2(1-|u|)^{p-2}|u|^2.$$

If now we suppose that R is an ordinary value of index α for $f(z)$ and take $N = N(R)$ we have, assuming

$$|z_0| \leqslant R\left(1+\frac{h}{N}\right), \qquad \left|\frac{z-z_0}{z_0}\right| = |u| < hN^{\frac{\alpha}{2}-1},$$

the inequality

$$|\chi(z, z_0)| < \sum_{1}^{\infty} C_{N+p}p^2(1+|u|)^p\left(R(1+\frac{h}{N})\right)^{N+p}$$
$$+ \sum_{1}^{N} C_{N-p}p^2(1-|u|)^{-p-2}\left(R(1+\frac{h}{N})\right)^{N-p}.$$

The hypothesis concerning $|u|$ enables us to apply the inequalities (4, 4) and (4, 5) to these two series and we see that

$$(4, 8) \quad |\chi(z, z_0)| < K\left(1+\frac{h}{N}\right)^N m(R)N^{3\left(1-\frac{\alpha}{2}\right)} < Km(R)N^{3-\frac{3}{2}\alpha},$$

for all sufficiently large values of R. Here K is independent of R. We have thus found an upper limit for the third term in equation (4, 6). A similar inequality for the second term can be deduced from (4, 6) itself. We put $|z|=|z_0|$ and $|z-z_0|/|z_0|=N^{\frac{3}{2}\left(\frac{\alpha}{2}-1\right)}$, so that the third term is less than KM(R), and it follows that

$$\text{(s)} \qquad N^{\frac{3}{2}\left(\frac{\alpha}{2}-1\right)}|g(z_0)| < KM(R)+2M(r_0).$$

Suppose that $R<r_0<R\left(1+\frac{h}{N}\right)$ and compare M(R) and $M(r_0)$; if z_0 is a point on the circle of radius r_0 such that $|f(z_0)|=M(r_0)$ and if $z/z_0=R/r_0$ we have, by (4, 6) and (s),

$$\left|f\left(z_0\frac{R}{r_0}\right)\right| > \left(1+\frac{h}{N}\right)^{-N}\left[M(r_0)-K(2+K)M(r_0)N^{\frac{1}{2}\left(1-3\frac{\alpha}{2}\right)}-KM(r_0)N^{1-\frac{3}{2}\alpha}\right].$$

If we take $\alpha>2/3$ the powers of N occuring in the bracket are negative and we have, for all sufficiently large values of R,

$$M(R)\geqslant\left|f\left(z_0\frac{R}{r_0}\right)\right|>\frac{1}{2}e^{-h}M(r_0).$$

The inequality (s) consequently becomes

$$(4,9) \qquad |g(z_0)|<KM(R)N^{3/2\left(1-\frac{\alpha}{2}\right)} \qquad |z_0|<R\left(1+\frac{h}{N}\right).$$

Now let z_0 be a point of the circle $|z|=R$ such that

$$|f(z_0)|>M(R)N^{-\beta}, \qquad (\beta>0)$$

and consider the region D_{z_0} in the plane of $z=re^{i\varphi}$, defined by the inequalities

$$|r-R|\leqslant\frac{h}{N}R, \qquad |\varphi-\varphi_0|N^{\beta'}\leqslant 1 \qquad \left(1-\frac{\alpha}{2}<\beta'<1\right).$$

If z is another point of the region D_{z_0} then

$$|f(z)| = K|f(z_0)|\left[1-\lambda\left(N^{3/2\left(1-\frac{\alpha}{2}\right)-\beta'+\beta}+N^{3\left(1-\frac{\alpha}{2}\right)-2\beta'+\beta}\right)\right],$$

where $|\log K| \leqslant h$ and λ is always finite. Therefore the ratio $|f(z)/f(z_0)|$ is confined between fixed positive limits, provided

$$\gamma = \beta' - \frac{3}{2}\left(1-\frac{\alpha}{2}\right) - \beta > 0.$$

If z_1 is another point of the region we have

$$f(z) = \left(\frac{z}{z_1}\right)^N f(z_1)(1+\eta_1(z, z_1)) \qquad (|\eta_1(z, z_1)| < KN^{-\gamma});$$

and, with the further condition

$$\left|\frac{z-z_1}{R}\right| < \frac{\varepsilon(R)}{M(R)} N^{-\frac{3}{2}\left(1-\frac{\alpha}{2}\right)},$$

this equation becomes

$$f(z) = \left(\frac{z}{z_1}\right)^N f(z_1) + \eta_1(z, z_1), \qquad |\eta_1(z, z_1)| < K\varepsilon(R).$$

To obtain complete precision we may give to α a numerical value. Putting $\alpha = 11/12$, $\beta = 1/16$ and $\beta' = 15/16$ we have $\gamma = 1/16$. The results obtained are collected in the following theorem :

THEOREM 29. — *Let R be a large ordinary value of index 11/12, h a given number, z_0 a point of modulus R such that*

$$(4, 10) \qquad |f(z_0)| > M(R)N^{-1/16} \qquad (N = N(R))$$

and D_{z_0} the region defined by the inequalities

$$|r - R| \leqslant \frac{h}{N} R, \qquad |\varphi - \varphi_0| < N^{-15/16};$$

(1) *If z is any point of* D_{z_0}, *then*

(4, 11) $$1/K < |f(z)/f(z_0)| < K$$

(2) *If z and z_1 are two points of* D_{z_0}, *then*

(4, 12) $$\left|\left(\frac{z_1}{z}\right)^N \frac{f(z)}{f(z_1)} - 1\right| < KN^{-1/16}.$$

(3) *If z and z_1 are points of* D_{z_0} *and subject to the further condition*

$$|z - z_1| M(R) < \varepsilon(R) R N^{-13/16},$$

then

(4, 13) $$\left|f(z) - \left(\frac{z}{z_1}\right)^N f(z_1)\right| < K\varepsilon(R).$$

(4) *The inequality*

(4, 14) $$|g(z)| = |zf'(z) - Nf(z)| < KM(R)N^{13/16}$$

is satisfied for all values of z of modulus less than $R\left(1 + \frac{h}{N}\right)$.

3. Theorems concerning $A(r)$ **and** $M'(r)$. — It follows from equation (4, 12) that for any ordinary value of R,

$$f(Re^{i\varphi}) = e^{i(\varphi-\varphi_0)N} f(z_0)(1 + \eta), \qquad |\eta| < KN^{-1/16}$$

provided $f(z_0)$ satisfies the condition (4, 10) and $|\varphi - \varphi_0| < N^{-15/16}$.

The argument $N(\varphi - \varphi_0)$ can therefore vary in an interval of length $N^{1/16}$, and consequently the imaginary part of $f(Re^{i\varphi})$ vanishes at $N^{1/16}/\pi$ points in the neighbourhood of the point z_0, the value of the real part at these points being equal to $\pm|f(z_0)|(1 + \eta)$, $|\eta| < KN^{-1/16}$.

Hence *for all ordinary values of* R

$$A(R) \backsim B(R) \backsim M(R).$$

In a similar manner we deduce from (4, 14) results concerning the derivative We have

$$\left|z\frac{f'(z)}{f(z)} - N\right| < KN^{13/16}\frac{M(R)}{|f(z)|}$$

and, in particular, if z, ($|z| = R$), is such that

$$|f(z)| > M(R)N^{-1/8},$$

this leads to

$$f'(z)/f(z) = (1 + \eta(z))\,N(R)/z.$$

Therefore $M'(R)$ is at least equal to $NM(R)(1 - \varepsilon(R))/R$. If z_1 is a point of modulus R at which $|f'(z_1)|$ is equal to its maximum $M'(R)$ we have, again applying equation (4, 14),

$$\left|z_1 - N\frac{f(z_1)}{f'(z_1)}\right| < KRN^{-3/16}.$$

So

$$Nf(z_1)/f'(z_1) = z_1(1 + \eta(z_1));$$

and the complementary inequality

$$NM(R) > RM'(R)(1 - \varepsilon(R))$$

is also satisfied. The following proposition in thus proved :

For all ordinary values of r

$$rM'(r) \backsim N(r)M(R),$$

and in the neighbourhood of points at which $|f(z)|$ *or* $|N(r)f'(z)/z|$ *is greater than* $M(r)N^{-1/8}$ *we have*

$$f'(z)/f(z) = (1 + hN^{-1/16})N(r)/z, \qquad |h| < K.$$

There is clearly a relation of the same form between the first and second derivatives, where $N(r)$ is replaced by the rank of the maximum term of $f'(z)$. It can be shewn that the rank of the maximum term of $f'(z)$ is asymptotically equivalent to $N(r)$ as r tends to infinity by ordinary values common to $f(z)$ and $zf'(z)$, the comparison functions being $\mathcal{F}(u)$ and $u\mathcal{F}'(u)$ respectively. Such common values certainly exist, for it is shewn by the theorem of § (IV, I) that the ratio of the total length of exceptional segments between 0 and R to R tends to zero.

In the first place it is clear, bearing in mind the form of $\mathfrak{F}(u)$, that the function $u\mathfrak{F}'(u)$ gives rise to a Newton's polygon with a vertex corresponding to every integral abscissa. For a given value u the ranks of the maximum terms of $\mathfrak{F}(u)$ and $u\mathfrak{F}'(u)$ are given by the equations

$$\log u + \alpha x^{\alpha-1} = 0 \qquad \text{and} \qquad \log u + \alpha x_1^{\alpha-1} + \frac{1}{x_1} = 0$$

respectively. Hence

$$x = x_1\left(1 + \frac{1}{\alpha x^{\alpha}}\right)^{\frac{1}{\alpha-1}}$$

So that, if N_1 is the rank of the maximum term in the derivative, then

$$\text{(s)} \qquad N_1 \geqslant N, \qquad N_1 < N(1 + KN^{-\alpha}).$$

Now let r be an ordinary value common to $f(z)$ and $zf'(z)$; we can find k and l such that the polygons $\pi(f)$ and $\pi[k\mathfrak{F}(r/l)]$ have the vertex of abscissa $N(r)$ in common. Consider the two functions $kr\mathfrak{F}'(r/l)/l$ and $zf'(z)$. Plainly the terms of rank $N(r)$ in these functions are equal and the first dominates the second. $\pi(zf')$ thus lies above $\pi_1 = \pi[kr\mathfrak{F}'(r/l)/l]$ and has in common with it the vertex of rank $N(r)$. The abscissa N_1 of the point of contact of π_1 and its tangent of slope $\log r$ is connected with $N = N(r)$ by the inequalities (s). Now the rank $N'(r)$ of the maximum term of $zf'(z)$ is equal to the abscissa of the point of contact of the tangent of slope $\log r$ with $\pi(zf')$ and is therefore also at least equal to N. We can find a polygon $\pi(k'r\mathfrak{F}'(r/l')/l')$ which does not lie above $\pi(zf')$ and has in common with it this vertex of abscissa $N' \geqslant N$, such that the line of slope $\log r$ through this point is a tangent of both polygons and such that $l' \geqslant l$. Therefore $r/l' \leqslant r/l$. Hence $N' \leqslant N_1$, and we have this result : *If r is an ordinary value common to $f(z)$ and $zf'(z)$*, ($zf'(z)$ compared with $u\mathfrak{F}'(u)$), *then the ratio of the ranks of the respective maximum terms is*

$$\frac{N'(r)}{N(r)} = 1 + \frac{h}{N^{\alpha}} \qquad 0 < h < K.$$

The same argument can obviously be applied to derivatives of successively higher order, and finally we obtain

THEOREM 30. — *r lies outside a set of exceptional segments in which, for $r > R$, the variation of $\log r$ is less than $KN(R/k)^{-1/12}$. Then at all points of the circle $|z| = r$ at which one of the numbers*

$$f(z),\ (z/N)f'(z),\ \ldots,\ (z/N)^q f^{(q)}(z),$$

is greater in modulus than $M(r)\,N(r)^{-1/8}$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{N(r)}{z}\right)^j (1 + h_j N^{-1/16}) \qquad (|h_j| < K),$$

and in particular

$$M^j(r) \backsim M(r)\,(N(r)/r)^j.$$

4. A relation between $M(r)$ and $m(r)$. — It is easy to find an asymptotic value for the ratio of $\mathfrak{F}_0(u)$ to its maximum term $m(u)$. It follows from the formulae (4, 4) and (4, 5), putting $q = 0$, $v = 0$, that

$$\mathfrak{F}_\alpha(u) < KN^{1-\frac{\alpha}{2}} m(u),$$

provided only $0 < \alpha < 1$.

On the other hand

$$\log m(u) = N \log u + N^\alpha,$$

N and u being functionally related by the familiar equation

$$\log u + \alpha x^{\alpha-1} = 0.$$

Therefore

$$\mathfrak{F}_\alpha(u) < K[\log m(u)]^{\frac{1}{\alpha}-\frac{1}{2}} m(u).$$

If r is an ordinary value, $f(z)$ is dominated by $k\mathfrak{F}_\alpha(r/l)$ and its maximum term is equal to the maximum term $km(u)$ of this function. Since k increases indefinitely with r we have $m(u) < m(r)$ for all sufficiently large values of r. Now α may be taken as near to unity as we please, and consequently we have the following proposition :

Given an arbitrarily small positive ε *the inequalities*

$$M(r) < N(r)^{\frac{1}{2}+\varepsilon}\, m(r)$$

$$M(r) < [\log m(r)]^{\frac{1}{2}+\varepsilon}\, m(r)$$

are satisfied for all sufficiently large values of r *exterior to a set of segments in which the total variation of log* r *is less than* $KN(R/k)^{-\varepsilon/(2+\varepsilon)}$ *for* $r > R$. $(k > 1)$.

The results of this section complete those of chapter II, but they are no longer valid for all values of r. This restriction is especially noticeable in the case of the relations between $M(r)$ and $M^1(r)$ in which $N(r)$, which may be subject to sudden arbitrary variations, occurs as a factor. It should be observed that the method of chapter II is a special case of that used here since it really consisted in taking the series $1/(1-u)$ as comparison function.

II. — The solutions of certain algebraic differential equations.

Theorem 30 enables us to prove that every solution of a linear differential equation with rational coefficients or an algebraic differential equation of the first order is of perfectly regular growth in the neighbourhood of an isolated essential singularity. Subject to certain restrictions the property can be shewn to belong also to equations of order higher than the first.

5. Linear equations with rational coefficients. — Let us consider a linear differential equation of order p with rational coefficients. Such an equation is of the form

$$(4,15) \qquad P_p(z)y^{(p)} + P_{p-1}(z)y^{(p-1)} + \dots + P_0(z)y + P_{-1}(z) = 0,$$

where the functions $P_j(z)$ are polynomials. Suppose that this equation is satisfied by an integral function $f(z)$. Let r be an ordinary value of index α and $z = re^{i\varphi}$ a point at which $|f(z)| = M(r)$. Now

$M(r)$ increases more rapidly than r^k, and so, denoting by $a_j z^{m_j}$ the term in $P_j(z)$ of highest degree and dividing through equation (4, 15) by y, we have, in virtue of theorem 30, the equation

$$\sum_0^p a_j N^j z^{m_j - j}(1 + \eta_j(z)) = 0,$$

where

$$|\eta_j(z)| < K_1 N^{-\delta} + K_2 r^{-1} \qquad (\delta > 0).$$

Now consider the algebraic equation [1]

$$\sum_0^p (a_j + t_j) x^j z^{m_j - j} = 0,$$

where the numbers t_j are parameters ($t_j \equiv 0$ if $a_j = 0$). For large values of r the function $X(z)$ defined by this equation has p branches, which may or may not be distinct, given by p expressions of the form

$$X(z) \backsim Az^l.$$

Here l is a rational number whose positive values are at least equal to $1/p$, and A is an algebraic function of the coefficients $a_j + t_j$. Let B denote the value of A when all the numbers t_j are zero. Then it is known that

$$|A - B| < K \varepsilon^{\delta_1} \qquad (\delta_1 > 0)$$

if all the numbers t_j are less than ε. Applying these results from the theory of algebraic functions to the present case we see that if Bz^l is the first term of one of the expansions (valid for large values of z) of the function defined by the equation

$$\sum_0^p a_j X^j z^{m_j - j} = 0,$$

[1] See appendix A.

then to every ordinary value r there correspond two numbers $B(r)$, $l(r)$ such that

$$\lim_{r=\infty} \frac{N(r)}{B(r) r^{l(r)}} = 1 .$$

Now the numbers $B(r)$, $l(r)$ can only take p different pairs of values and $N(r)$ is an increasing function. It is plain that if the values of B and l change as r traverses any ordinary segment sufficiently distant from the origin the new values must be greater than the old. For if either B or l were to diminish $N(r)$ would shew a sudden decrease. Again if R and R' are the endpoints of an exceptional interval we have

$$N(R) = B(R) R^{l(R)}\left(1 + \eta(R)\right), \qquad N(R') = B(R') R'^{l(R')}\left(1 + \eta(R')\right)$$

But since R'/R tends to 1, we have also

$$N(R') = B(R') R^{l(R')}\left(1 + \eta(R')\right),$$

shewing that B and l cannot decrease as r traverses the segment (R, R'). These numbers B and l are therefore ultimately fixed and we have, for all sufficiently large ordinary values of r,

$$N(r) \backsim Br^{l},$$

where B is a constant and l a fixed rational number greater than or equal to $1/p$. And in view of the known density of the ordinary values (we had $\lim (R'/R) = 1$) this relation clearly holds for all values of r. The function is of finite order and

$$\log M(r) \backsim \int_{r_0}^{r} N(x)\, dx/x \backsim B \frac{1}{l} r^{l}.$$

An integral function satisfying a linear differential equation with rational coefficients is therefore of finite rational order and of perfectly regular growth.

This result can be generalized. Consider a solution of the form

$$y = z^{\mu} F(z) = z^{\mu} \Phi\left(\frac{1}{z}\right) f(z),$$

where μ may be any number and $F(z)$ has an isolated essential singularity at infinity. It is known that the equation (4, 15) has always at least one solution of this form, or one solution which is the sum of a function of this form and a function regular at infinity, unless all its solutions are regular at infinity (¹). We have

$$\frac{y'}{y} = \frac{f'(z)}{f(z)} + \frac{1}{z}\Phi_1\left(\frac{1}{z}\right) \qquad \left(\Phi_1\left(\frac{1}{z}\right) = b_1 + \frac{b_2}{z} + \dots\right)$$

and this ratio will therefore behave in the same way as that corresponding to an integral function. Again

$$\frac{y''}{y} = \frac{f''(z)}{f(z)} + \left(\frac{y'}{y}\right)^2 - \left(\frac{f'(z)}{f(z)}\right)^2 + \frac{1}{z^2}\Phi_2\left(\frac{1}{z}\right),$$

and consequently y''/y behaves like f''/f, and so on. The above result is therefore valid for solutions of this form also. *In every solution of the form*

$$y = z^\mu \Phi\left(\frac{1}{z}\right) f(z)$$

the function $f(z)$ is of finite rational order and of perfectly regular growth.

6. Algebraic differential equations. — The argument of the last paragraph can be easily extended to the case of algebraic differential equations of the first order. Such an equation may be written

$$(4, 16) \qquad G(y, zy'/y, z) = 0,$$

G being a polynomial in the variables in the bracket, of degree m in y. Let

$$\Psi(zy'/y, z) = \sum a_{p,q}(zy'/y)^p z^q \qquad (0 \leqslant p \leqslant P;\ 0 \leqslant q \leqslant Q)$$

be the coefficient of y^m in G. Suppose that equation (4, 16) has a

(¹) Regular in the sense in which the word is used in the theory of linear differential equations. See Whittaker and Watson : *Modern Analysis*, 2'nd edition, p. 191 et seq.

solution which is an integral function of z or, more generally, a function of the form considered above. Giving $z = re^{i\varphi}$ (r ordinary) a value such that $y = M(r)$ and dividing through by y^m, the equation becomes

$$\Psi(zy'/y, z) + G_1(1/y, zy'/y, z)/y = 0.$$

where G_1 is a polynomial. And since every expression of the form

$$\frac{1}{y^x}\left(\frac{zy'}{y}\right)^p z^q < 2\frac{N^p r^q}{M(r)^x}$$

is less than $(Nr)^{-K}$, for all values of K, this may be written

$$\sum a_{p,q}(1 + \varepsilon_{p,q}(z))\, N^p z^q = 0$$

which is an equation of the form considered in the last paragraph. We have once more therefore

$$N(r) \backsim Br^l,$$

where the second member is the first term of a certain expansion of the function $V(z)$ in the neighbouohood of infinity, $X(z)$ being defined by the equation

$$\sum a_{p,q} X^p z^q = 0.$$

The different values which may be assigned to l and B can be determined by Puiseux's method.

Functions of the form $z^\mu F(z)$ which satisfy an algebraic differential equation of the first order, $F(z)$ having an isolated essential singularity at infinity, are necessarily of finite rational order and of perfectly regular growth.

This result is not however true in general of equations of order higher than the first. For it may happen that if in such an equation $y^{(p)}$ is replaced by $y(N/z)^p$ certain terms are made to vanish. This occurs for instance in the case of the expression $yy'' - y'^2$, which vanishes identically. But it is clear that our conclusions remain unchanged if the terms of highest degree in y are unaffected by reductions

consequent on this substitution. Thus, *if, in each term,* $y^{(p)}$ *is replaced by* $y(N/z)^p$ *and if, after this substitution, none of the terms of highest degree in* y *disappear in consequence of reductions, then all solutions of the form* $z^\mu F(z)$ *are of finite rational order and of perfectly regular growth.*

The reader will observe that we have at the same time a condition for the existence of a solution of the form $z^\mu F(z)$. It is necessary that after the above substitution has been made the polynomial in N and z, which is the coefficient of the term of highest degree in y, should contain at least two different terms.

III. — The zeros of an integral function of infinite order.

Certain properties of the growth of $M(r)$ which enable us to extend theorem 25 to the case of functions of infinite order can be proved with the aid of the dominant of index α.

7. The growth of $\log M(r)$. — Consider the function $U_\alpha(r)$, *the dominant of index* α. § (IV. 1). If $m_\alpha(r)$ is the maximum term in this function and $V(r) = \log m_\alpha(r)$, then

$$\log U_\alpha(r) \backsim V(r),$$

since the inequalities of § (IV, 4) are everywhere valid. Now $m_\alpha(r)$ is equal to the maximum term of some function $k\mathfrak{F}_\alpha(r/l)$, and consequently

$$V(r) = k' + N^\alpha + N \log(r/l) \qquad (k' = \log k),$$

where the value of N differs by less than 1 from the root of the equation

$$\alpha x^{\alpha-1} = \log(l/r).$$

Hence

$$(1)\quad V(r) = k' + (1 + \eta_1(r))(1-\alpha)N^\alpha = k' + (1 + \eta_1(r))\, C[\log(l/r)]^{\alpha/(\alpha-1)}$$

$$C = (1-\alpha)\,\alpha^{\frac{\alpha}{\alpha-1}}.$$

But $k\mathfrak{F}_\alpha(r/l)$ dominates $U_\alpha(r)$, and so, in every interval $r < r' < l$, we have

$$V(r') < k' + (1 + \eta(r))\, C(\log(l/r'))^{\alpha/(\alpha-1)}$$

since l/r is near to 1 we may choose r' such that

$$\log(r'/r) < [\log(l/r)]^\delta \qquad (\delta > 1).$$

The substitution of this value r' for r in equation (t) will not appreciably affect the value of the right-hand side. We are thus led to take

$$\text{(w)} \qquad r' = r\left[1 + (V(r))^{\frac{\alpha-1}{\alpha}\delta}\right] \qquad (\delta > 1).$$

It follows from (t), in which $k' > 0$, that

$$V(r) < K[\log(l/r)]^{\frac{\alpha}{\alpha-1}}.$$

So the number r' defined by equation (w) is less than

$$r\left[1 + K\left(\log\frac{l}{r}\right)^\delta\right]$$

and

$$V(r') < k' + (1 + \eta(r))\, C\left(\log\frac{l}{r}\right)^{\frac{\alpha}{\alpha-1}} < (1 + \varepsilon(r))\, V(r).$$

Therefore the dominant $U_\alpha(r)$ satisfies the following condition :

$$(4, 17) \qquad \log U_\alpha(r') < (1 + \varepsilon(r)) \log U_\alpha(r)$$

where

$$(4, 18) \qquad r' = r\left[1 + K(\log U_\alpha(r))^{1 - \frac{1}{\alpha'}}\right] \qquad (\alpha' < \alpha).$$

For all ordinary values of index α the same inequality is satisfied by $\log M(r)$.

Borel has shewn that every increasing function $\Psi(x)$ satisfies some condition of this kind. If we put

$$x' = x + [\Psi(x)]^{-\delta} \qquad (\delta > 0)$$

we have, in general,

$$\lim_{x=\infty} \Psi(x')/\Psi(x) = 1 :$$

a certain set of segments, whose total length beyond a certain value X is of order $\Psi(X)^{-\delta}$, has to be excluded. The property of growth of log $M(r)$ which has just been established is thus not restricted to functions to which our argument applies and it would be possible to give a proof independent of the notion of an ordinary interval. But our method is interesting in that it connects the intervals in which log $M(r)$ is of *normal growth* with the intervals in which the inequalities of the section I are satisfied.

8. The zeros of functions of infinite order. — It was assumed in theorem 25 that the function was of finite order. This hypothesis was only used to find an upper limit for $|\Phi_1'(z)|$ in passing from the inequality (m) to the first inequality (o). The result of the last paragraph permits us to dispense with it. We had

$$A_1(R_6) < \left(1 + \frac{HH'}{\log k'}\right) \log M(R_7).$$

Assuming R_1 to be an ordinary value of index α (near to 1) and

$$R/R_1 = 1 + K(\log M(R_1))^{-\delta} \qquad \left(\delta = \frac{1}{\alpha'} - 1,\ \alpha' < \alpha\right),$$

which gives

$$k' = (R/R_1)^{1/7} = 1 + K(\log M(R_1))^{-\delta}$$
$$\log k' = K(\log M(R_1))^{-\delta}$$
$$H' = K \log_2 M(R_1),$$

the inequality (m) becomes

$$A_1(R_6) < KH(k)\,(\log M(R_1))^{1+2\delta},$$

It now follows from corollary (8) that, for $|z|^2 \leqslant R_5 R_6$,

$$|\Phi_1(z)| < \frac{4}{\sqrt{k'} - 1} A_1(R_6) < KH(\log M(R_4))^{1+3\delta}$$

and theorem 13 shews that, provided $|z| \leqslant R_5$, $|\Phi_1'(z)|$ is less than the product of this expression and $1/R_4(\sqrt{k'} - 1)$. Therefore since $H(k)$ is bounded and $\log M(R_4)$ is not, we have

$$|\Phi_1'(z)| < (\log M(R_4))^{1+4\delta}.$$

Consequently, if $H(k)$ is such that

$$(\log M(R_4))^{1+4\delta} < M(R_4)^{H(k)},$$

the argument of theorem 25 is valid from this point onwards. Thus, if $H(k)$ satisfies this condition together with the last inequality of § 12, which becomes, on replacing k' and H' by their values,

$$KH(k)(\log M(R_4))^{1\delta} < 1,$$

then the relation

$$\int_\beta^R \frac{n(x, a) + n(x, b)}{x} dx < H(k) \log M(R_4)$$

is impossible. The two conditions will clearly be satisfied if we put

$$H(k) = [\log M(R_4)]^{-4\delta}.$$

If R_4 *is an ordinary value of index* α *and* R *is equal to*

$$R_4[1 + K(\log M(R_4))^{-\delta}] \qquad \delta > \frac{1}{\alpha} - 1,$$

then

$$\int_\beta^R \frac{n(x, a) + n(x, b)}{x} dx > [\log M(R)]^{1-4\delta}.$$

As α may be taken as near to 1 as we please the following proposition is proved :

THEOREM 31. — *If $f(z)$ is an integral function of infinite order and a and b are two different numbers, then, given any ε, the number $n(x, a, b)$ of zeros of the function $(f(z) - a)\,(f(z) - b)$ satisfies the inequality*

$$\int_{\beta}^{r} n(x, a, b)\,dx/x > (\log M(r))^{1-\varepsilon} \qquad (\beta > 0)$$

for all values of r, save possibly in a set of intervals in which the total variation of $\log r$ *is bounded.*

It is possible to deduce from this theorem an inequality which, though in some cases it may be lacking in precision, is valid for all sufficiently large values of r. In fact, bearing in mind the characteristic property of exceptional intervals $[(R, R')$ being such an interval $\lim R'/R = 1]$, we see, since the integral is an increasing function, that *given any ε and any $k > 1$ the inequality*

$$\int_{\beta}^{r} n(x, a, b)\,dx/x > [\log M(r/k)]^{1-\varepsilon}$$

is satisfied for all sufficiently large values of r.

It is moreover clear that in those intervals in which $\log M(r)$ is less than r^h, h being any fixed positive number, theorem 25 applies and the number ε may be suppressed.

We have thus obtained a precise result complementary to Picard's theorem and independent of any consideration of order.

IV. — A DIRECT PROOF OF THE GENERAL PICARD THEOREM.

We have investigated the behaviour of $f(z)$ in the neighbourhood of its points of maximum modulus and on the results obtained we shall found a proof of the general Picard theorem : *A function $F(z)$, having an isolated essential singularity at infinity, assumes every value an infinity of times in the neighbourhood of the point at infinity, save possibly a single exceptional value.* The argument is a direct one. It consists in shewing that if $F(z)$ has no zeros outside a certain circle

described about the origin, then this function assumes every value $a \neq 0$ in all the regions D_{z_0} of theorem 29 which are sufficiently distant from the origin.

9. Picard's theorem. — A function $F(z)$ having no zeros outside a circle $|z| = R_0$ is of the form

$$F(z) = z^{\mu} e^{f(z)+\Psi(z)},$$

where $f(z)$ is an integral function and $\Psi(z)$ is regular for $|z| > R_0$ and zero at infinity. The equation

$$(4, 19) \qquad F(z) = a$$

may therefore be written

$$(4, 20) \quad f(z) + \mu \log z - \log |a| - 2i\pi(\theta + q) + \Psi(z) = 0,$$

$2\pi\theta$ being the argument (lying between 0 and 2π) of a, q a positive or negative integer or zero, and $\log z$ one of the determinations of the logarithmic function. To each couple of numbers q and z satisfying equation (4, 20) there corresponds a solution of (4, 19).

Now suppose that a lies between two numbers $1/A$, A, where A is at our disposal, and that z lies in a region D_{z_0} sufficiently distant from the origin to ensure that R exceed any constants which may appear and that $M(R)$ be greater than R. We take for $\log z$ that determination which is continuous and has its imaginary part less in absolute value than 2π in D_{z_0}.

Equation (4, 12), written

$$f(z) = \left(\frac{z}{z_1}\right)^N f(z_1)\,(1 + \eta_1(z)), \qquad N = N(r)$$

shews that on every arc $|z| = \text{constant}$, interior to D_{z_0}, the argument of $f(z) + \mu \log z$, which is continuous, varies by more than $N^{1/16}$, so that on this arc there are more than $\frac{1}{\pi} N^{1/16}$ points z_1 at which the real part of this quantity vanishes. z_1 being such a point and $2i\pi Q$ the value of $f(z_1) + \mu \log z_1$ we have

$$(\xi) \qquad |Q| > KM(R)N^{-1/16}.$$

Let Q_1 be the greatest integer less than Q and put $q = Q_1$. Equation (4, 20) may be written in the form

$$G(z) + H(z) = 0,$$

where

$$G(z) = 2i\pi Q\left(\frac{z}{z_1}\right)^N - \log|a| - 2i\pi(\theta + Q_1),$$

$$H(z) = f(z) - \left(\frac{z}{z_1}\right)^N f(z_1) + \mu\left[\log z - \left(\frac{z}{z_1}\right)^N \log z_1\right] + \Psi(z).$$

Now the function $G(z)$ vanishes for

$$\frac{z}{z_1} = \left[\frac{\log|a| + 2i\pi(\theta + Q_1)}{2i\pi Q}\right]^{1/N} = \left[1 + \frac{\log|a| + 2i\pi(\theta + Q_1 - Q)}{2i\pi Q}\right]^{1/N}$$

and, since $|\theta'| = |\theta + Q_1 - Q| < 2$, the last term in the last bracket is less in modulus than $(\log A + 5\pi)/2\pi Q = K/Q$, which in virtue of ($\xi$) is very small. Therefore

$$z = z_1\left(1 + \frac{\log|a| + 2\pi\theta'}{2i\pi QN} + \frac{K}{NQ^2}\right).$$

The equatión $G(z) = 0$ has a root in the circle C_{z_1} defined by the equation

$$z = z_1\left(1 + \frac{\log|a| + 2\pi\theta'}{2i\pi QN}\right) + \gamma\frac{z_1}{QN}e^{ix}$$

where γ is an arbitrary fixed number and x is real and varies from 0 to 2π. On the circumference of this circle we have

$$G(z) = 2i\pi Q\left(1 + \frac{\log|a| + 2\pi\theta'}{2i\pi Q} + \gamma\frac{e^{ix}}{Q} + \frac{K}{Q^2}\right) - \log|a| - 2i\pi(\theta + Q_1)$$

or

$$G(z) = 2i\pi\gamma e^{ix} + K/Q,$$

and so

$$|G(z)| > \pi\gamma.$$

Now consider the function $H(z)$. For points on the circumference of C_{z_1} we have

$$|z - z_1| < \frac{KR}{QN} < \frac{KR}{M(R)} N^{-15/16},$$

and hence, by the third part of theorem 29, where we put $\varepsilon(R) = KN^{-1/8}$,

$$\left| f(z) - \left(\frac{z}{z_1}\right)^N f(z_1) \right| < KN^{-1/8}.$$

In addition, the difference $\log z - (z/z_1)^N \log z_1$ may be written

$$\log\left(\frac{z}{z_1}\right) - \left[\left(1 + \frac{z - z_1}{z_1}\right)^N - 1\right] \log z_1$$

and this is clearly less in modulus than $K \log R/Q$. Hence on the circumference C_{z_1}, $H(z)$ tends to zero with $1/R$. Thus finally on this circumference

$$|H(z)/G(z)| < 1.$$

Therefore, since the functions $H(z)$ and $G(z)$ are regular in the circle C_{z_1}, the number of roots of the equation

$$G(z) + H(z) = 0$$

in this circle is equal to the number of roots of the equation $G(z) = 0$ (Rouché's theorem) (1) :. there is at least one root of eqūation (4, 19) in C_{z_1}. This clearly proves the theorem.

In addition to proving Picard's theorem the resources of this method enable us to make deductions about the distribution of the zeros of $F(z) - a$.

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(1) Rouché's theorem : *The functions $f(z)$ and $\Phi(z)$ are regular in a region bounded by a closed curve C, and on C we have $|\Phi(z)/f(z)| < 1$. Then the number of zeros of $f(z) + \Phi(z)$ in C is equal to the number of zeros of $f(z)$ in C.* The proof is very simple. It is known that if $f(z)$ has K zeros in C, then as z describes the path C, the change in the argument of $f(z)$ is equal to $2\pi K$. Now $f(z) + \Phi(z) = f(z)\,(1 + (\Phi(z)/f(z)))$ and, since by hypothesis $|\Phi/f| < 1$ on C, it is clear that the change in the argument of the last factor as z describes C is zero. Therefore the change in argument of $f + \Phi$ is equal to $2\pi K$, and the theorem is proved.

The circle C_{z_1} lies inside a circle of centre z_1 and radius not greater than

$$R\frac{\log A + 4\pi + 2\pi\gamma}{2\pi QN} < \frac{KR}{N\,|f(z_0)|} = d$$

and on each of the arcs

$$(\sigma) \qquad |z| = R + 2\lambda d \qquad (\lambda = 0, \pm 1, \ldots \pm E(R/2dN))$$

interior to D_{z_0}, there are $\frac{1}{\pi}N^{1/16}$ points z_1 to each of which the above argument applies. Each of them is the centre of a circle of radius d containing a zero of $F(z) - a$, where $1/A \leqslant |a| \leqslant A$. Now these circles do not cut one another. For the distance separating two points z_1 on the same arc (σ) is equal to $2\pi(1+\eta(R))R/N > 2d$. Circles with their centres on different arcs (σ) cannot cut, by definition. We have thus found $KN^{1/16}|f(z_0)|$ zeros of $F(z) - a$ in the region D_{z_0}. In particular this is true of the region D_{z_0} corresponding to the point z_0 where $|f(z_0)| = M(R)$, and we may state the following proposition :

THEOREM 32. — *The number of roots of equation (4, 19) in every region* D_{z_0} *where* $|f(z_0)| = M(R)$, *sufficiently distant from the origin, exceeds* $KN^{1/16}M(R)$. *In the region* D_{z_0} *we can describe a set of non-overlapping circles of radius* $K'R/NM(R)$ *whose centres lie at the vertices of a network of curvilinear quadrilaterals, such that each circle contains a zero of every function* $F(z) - a$, *where* $|a|$ *lies between two assigned positive numbers.*

In order to remove the restriction on the value of a the circles of radius d can be replaced by circles of radius $\chi(R)R/NM(R)$, $\chi(R)$ being a very slowly increasing function. The number of such circles in D_{z_0} is now only $KN^{1/16}M(R)/\chi(R)$, but we can assert that for $R > R_a$ every function $F(z) - a$ has a zero in each of them.

REFERENCES

§ 1, 2, 3 and 4 : Wiman **5** and **6**; Valiron **7** and **8**.

§ 5 and 6 : Wiman **6**; Valiron **14**.

§ 7 and 8 : Borel **2**; Valiron **8** and **13**.

§ 9. Valiron **8**.

CHAPTER V

Asymptotic values and paths of determination.

It follows from a consideration of the properties of functions of the form $e^{Q(z)}$, where $Q(z)$ is a polynomial, that an integral function, or a function $F(z)$, may tend steadily to zero as z tends to infinity by paths interior to certain angles. If we can find an angle in which $F(z)$ tends to a definite limit the function may be regarded as being completely known, and the essential singularity as being without influence, in that angle. There are also important deductions to be made concerning the inverse function (¹) of $F(z)$. In this chapter we shall restrict ourselves to an investigation of the properties of the asymptotic values themselves, without touching on these collateral questions, though their importance is obvious enough.

It was shewn by Wiman that functions of order less than $1/2$ have no finite limiting values. On every path proceeding to infinity the upper limit of $|F(z)|$ is equal to $+\infty$. Our proof of this theorem is based upon a result which is now classical, Phragmèn and Lindelöf's generalisation of Cauchy's theorem on the modulus of a regular function. It is a generalisation of which many beautiful applications have been made, notably by Lindelöf himself. Functions of order less than $1/2$ are thus to some extent analogous to polynomials and we shall shew that in the case of functions of zero order such that the ratio

$$\log M(r)/(\log r)^2$$

is bounded the analogy with polynomials is even closer. For if the zeros are excluded by certain small regions the logarithm of the modulus becomes asymptotically equivalent to the logarithm of its maximum as z recedes (²) along any path in the remainder of the plane.

(¹). See Appendix D.

(²) As we shall often have occasion in this chapter to speak of continuous paths proceeding from a point A to infinity we propose, for the sake of brevity to call them *receding paths*. When z traverses such a path we shall say that *z recedes*.

The rest of the chapter is devoted to the study of the paths along which $F(z)$ tends to its asymptotic values. The principle result established is a theorem recently proved by Carleman, who has shewn that for a function of order ρ there cannot be more than 5ρ non-contiguous assemblages of such paths.

I. — Lindelöf's theorem and functions of order less than 1/2.

1. A generalisation of a theorem of Cauchy. — Let us consider a closed region D bounded by a contour Γ formed by one or more arcs of simple curves. We know that the modulus of a function $\Phi(z)$ regular in D attains its maximum value at a point of the contour. It is moreover sufficient for the truth of the theorem that the function should be analytic and regular and of uniform modulus in D. Thus if the function is bounded on the contour Γ, $|\Phi(z)| < K$, it is also bounded at all points interior to D.

Now suppose that $\Phi(z)$ is analytic and regular at all points of D except for a finite number of points $A_1, A_2, \ldots, A_m$, of the contour Γ and that on the intervening open arcs of Γ its modulus is less than or equal to K, this modulus being uniform in D. With these conditions the original form of Cauchy's theorem no longer applies. But suppose that there is a function $\omega(z)$ regular in the same domain as $\Phi(z)$, having no zeros interior to D and of modulus less than 1 on the various open arcs of Γ determined by the points A_j. If further, for all positive values of α, the product

$$\Psi(z, \alpha) = \Phi(z)\,[\omega(z)]^{\alpha}$$

tends uniformly to zero as z approaches any of the points A_j by a path interior to D, the function $\Psi(z, \alpha)$ is again a function whose modulus does not exceed K on the arcs of Γ. And if z_0 is an interior point of D we can, for all α, describe a circle about each of the points A_j of radius such that z_0 lies in the open domain Δ_α formed by the points interior to D and exterior to these small circles and such that $|\Psi(z, \alpha)| < K$ on the closed arcs of these circles belonging

to D. Thus $|\Psi(z, \alpha)| \leqslant K$ on the boundary of Δ_ϵ and so, by Cauchy's theorem, we have for z_0

$$|\Psi(z_0, \alpha)| = |\Phi(z_0)| \quad |\omega(z_0)|^\alpha \leqslant K.$$

Hence, since $\omega(z_0) \neq 0$,

$$|\Phi(z_0)| < K\,|\omega(z_0)|^{-\alpha}.$$

But this inequality is true for all α and the value of $\Phi(z_0)$ is independent of α. We therefore conclude that

$$|\Phi(z_0)| \leqslant K.$$

THEOREM 33 (LINDELÖF'S THEOREM). — *Suppose that the function $\Phi(z)$ is analytic, regular and of uniform modulus in a finite, open, connected domain D, and regular and of modulus less than K on the various open arcs of the bounding contour Γ determined by a finite number of points A_j. Suppose further that there exists another function $\omega(z)$ satisfying the same conditions with the additional provisions that it does not vanish in D and that $|\omega(z)| < 1$ on these arcs of Γ, such that the product $\Phi(z)[\omega(z)]^\alpha$ tends uniformly to zero, for all $\alpha > 0$, as z approaches any one of the points A_j by a path interior to D. Then we can assert that*

$$|\Phi(z)| \leqslant K$$

at all interior points of D.

We may observe that it is permissible to suppose that certain portions of the contour Γ are frontiers of contiguous portions of D. For example, Γ might consist of a circle and a radius.

As a preliminary application of the theorem let us assume that $|\Phi(z)|$ is bounded in D. We may take

$$\omega(z) = R^{-m}(z - a_1)(z - a_2) \ldots (z - a_m),$$

where $z = a_j$ at the point A_j and R is the maximum diameter of D; this function satisfies all the conditions imposed in theorem 33, and we can deduce the following corollary :

COROLLARY 33. — *Let $\Phi(z)$ be a function regular in a finite do-*

main D *and on its contour save at certain points* $A_1, A_2, \ldots, A_m$, *and let* $|\Phi(z)| \leqslant K$ *on the arcs of the contour determined by these points* A_j. *Then, either* $|\Phi(z)| \leqslant K$ *at all points of* D, *or there exists at least one sequence of points of* D, *with one or more of the points* A_j *as limit, such that in this sequence*

$$\lim |\Phi(z)| = \infty.$$

By means of this corollary we are able to shew that for every function $F(z)$, with an essential singularity at infinity, there exist paths along which the modulus of the function tends to infinity.

In the first place we have seen that the maximum modulus $M_1(r)$ of $F(z)$ is ultimately an increasing function. Let $R(>1)$ be a number such that $M_1(r)$ is an increasing function for all $r > R$ and let us consider an indefinitely increasing sequence of numbers

$$X_1 = M_1(R), X_2, \ldots\ldots, X_m, \ldots.$$

Now it follows from Liouville's theorem that there is a point outside the circle $|z| = R$ at which $|F(z)| > X_1$, and it is clear that the set of points at which $|F(z)| > X_1$ constitute a set of one or more connected open domains whose contours are continuous curves on which $|F(z)| = X_1$. One of these domains is clearly exterior to the circle $|z| = R$. Let us denote it by D_1. Now this domain is infinite. For if it were finite we should have $|F| = X_1$ on the boundary and $|F| > X_1$ at interior points, in contradiction with Cauchy's theorem. Moreover $|F|$ is unbounded in the domain, for otherwise we should have a contradiction of corollary 33. There is therefore a point of D_1 at which $|F(z)| > X_2$, and consequently a domain D_2, interior to D_1, such that at all points of D_2, $|F| > X_2$. This argument may be repeated for $X_3, \ldots, X_m, \ldots$ and so on. We have thus found a sequence of infinite domains $D_1, D_2, \ldots, D_m, \ldots$, each interior to the preceding one and such that in D_m we have $|F| > X_m$, whilst on the contour, which is a continuous curve, $|F| = X_m$. If now we join a point of the contour of D_1 to a point of the contour of D_2 by a continuous curve lying in D_1, then this point of the contour of D_2 to a point of the contour of D_3 by a continuous curve lying in D_2, and repeat the process for each domain, we shall obtain a continuous receding path along which $F(z)$ tends to infinity.

In general we shall say that a receding path along which $F(z)$ tends to a value ω is *a path of determination* ω and that ω *is an asymptotic value*. The result we have proved can then be stated in the following form :

For every function $F(z)$ *there exist paths of determination* ∞, or, *infinity is an asymptotic value of* $F(z)$.

If $F(z)$ has a value exceptional P, say b, then the function $\frac{1}{F(z)-b}$ has an isolated essential singularity at infinity, and so has paths of determination ∞. Therefore if b *is a value exceptional* P *for* $F(z)$, *there is a path of determination* b *for this function.*

2. Wiman's theorem. The following result, due to Phragmèn, can be deduced from theorem 33.

THEOREM 34. — *Let* $\Phi(z)$ *be a function regular in the open domain*

$$r > R, \qquad |\varphi| < \frac{\pi}{2\gamma} \qquad \left(\gamma \geqslant \frac{1}{2}\right)$$

and on the finite part of the contour, and suppose that the function is of order less than γ *in this angle* . i. e . *there is a number* γ' *less than* γ *such that*

$$\log|\Phi(z)| < r^{\gamma'} \qquad (r = |z| > R', \ |\varphi| < \pi/2\gamma).$$

Then if $|\Phi(z)| < K$ *on the contour of the domain, the same inequality holds good throughout the domain.*

The proof is immediate. Let us take

$$\omega(z) = e^{-z^{\gamma''}} \qquad (\gamma' < \gamma'' < \gamma),$$

the determination of $z^{\gamma''}$ being such that the function is real and positive on the positive real axis. Then we have in the domain and on its contour

$$\log|\omega(z)| = -r^{\gamma''}\cos(\gamma''\varphi) < -\beta r^{\gamma''} \qquad (\beta > 0).$$

$\omega(z)$ does not vanish in the domain, $|\omega(z)| < 1$ on the contour and,

as z recedes by a path in the domain, the product $[\omega(z)]^a\Phi(z)$ tends uniformly to zero. For

$$\log[|\Phi(z)|\,|\omega(z)|^a] < -\alpha\beta r^{\gamma''} + r^{\gamma'} \qquad (r > R')$$

and

$$\gamma'' > \gamma'.$$

The result then follows from theorem 33.

In particular if $F(z)$ has an isolated essential singularity at infinity and is of order $\rho < 1/2$, there is no line $\varphi =$ constant on which $F(z)$ is bounded. For suppose the contrary to be the case and that $F(z)$ is bounded along a straight line, which it is plainly permissible to regard as coincident with the negative real axis. Then, applying the foregoing theorem with $\gamma = 1/2$, $\gamma' = \rho + \varepsilon < 1/2$, we should conclude that $F(z)$ is bounded in the whole plane $|z| > R$, which is impossible.

In general if $F(z)$ is of order $\rho < 1/2$ we have

$$F(z) = z^\tau \Phi\left(\frac{1}{z}\right) f(z), \qquad f(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right),$$

where $\Phi\left(\frac{1}{z}\right)$ assumes a finite value A at infinity. Provided that r is sufficiently large the minimum modulus of $F(z)$ on the circle $|z| = r$ is greater than

$$K r^\tau \prod \left(1 - \frac{r}{r_n}\right)$$

and therefore greater than the modulus of

$$F_1(z) = K z^\tau \prod_1^\infty \left(1 + \frac{z}{r_n}\right)$$

at the point of modulus r on the negative real axis. But, since the moduli of its zeros are equal to the moduli of the zeros of $F(z)$, $F_1(z)$ is also of order $\rho < 1/2$ and is therefore unbounded on the negative real axis. Hence we have :

THEOREM 35. — *Given a function* F(z) *of order less than* 1/2 *we can find a sequence of circles of indefinitely increasing radii described about the origin as centre on which the minimum modulus of* F(z) *tends to infinity. There is therefore no finite asymptotic value for functions of order less than* 1/2.

Here we have an analogy between a function F(z) of order less than 1/2 and a function with a pole at infinity. A more precise investigation of the minimum modulus on certain circles shews however that the analogy is even closer than is suggested by theorem 35. In view of the preceding argument we may clearly confine ourselves to the case of an integral function equal to 1 at the origin and having its zeros on the negative real axis. It is plain that if we are given an integral function equal to 1 at the origin and if we change the arguments of its zeros without changing the moduli so as to bring them all into line we shall thereby increase the maximum and diminish the minimum modulus on any circle $|z| = r$. So that any inequalities found to hold between the maximum and minimum modulus in this particular case will hold in general.

Now suppose that $\rho(r)$ is a proximate order L of $f(z)$ subject to the conditions of § (III, 6) : log M(r) satisfies the condition (3, 13)

$$\overline{\lim_{r=\infty}} \frac{\log M(r)}{r^{\rho(r)}} = 1.$$

Let V(z) be a function analytic and regular in the domain

$$r > R, \qquad -\pi \leqslant \varphi \leqslant \pi,$$

and regular on the finite part of the contour, such that

$$V(re^{i\varphi}) \backsim r^{\rho(r)} e^{i\varphi\rho},$$

and suppose that V(z) is real on the positive real axis.

Now the function

$$f(z)\, e^{-kV(z)}$$

is regular and of order less than 1/2 in the domain D. Its modulus

will therefore be bounded in D if it is bounded on the contour, and in particular we shall have

$$\log f(r) < k\mathrm{V}(r) + \mathrm{K},$$

and if $k < 1$ this leads to a contradiction. Therefore, for an indefinitely increasing sequence of values of r,

$$\log |f(-r)| > k_1 \cos(\pi\rho) r^{\rho(r)},$$

provided $k_1 < 1$; and Wimans's theorem may be stated in the following precise form :

THEOREM 36. — *A function* F(z) *of order* $\rho < 1/2$ *and of proximate order* L *equal to* $\rho(r)$ *satisfies the inequality*

$$\log |f(z)| > [\cos(\pi\rho) - \varepsilon]\, r^{\rho(r)} \qquad (\varepsilon > 0) \tag{5, 1}$$

on an infinite sequence of circles of infinitely increasing radius. We have, a fortiori, on these circles

$$\log |f(z)| > [\cos(\pi\rho) - \varepsilon] \log \mathrm{M}(r). \tag{5, 2}$$

It remains to establish the existence of the function V(z). We start from a canonical product

$$\mathrm{P}(z) = \prod_{1}^{\infty} \left(1 + \frac{z}{r_n}\right)$$

of order less than 1 with its zeros situated on the negative real axis such that (¹)

$$n(r) = [r^{\rho_1(r)}],$$

where $\rho_1(r)$ is a proximate order L and tends to the value ρ_1 as r tends to infinity. So that, however small δ may be,

$$r^{\rho_1(r) - \rho_1 + \delta}, \qquad r^{\rho_1 + \delta - \rho_1(r)}$$

are increasing functions for $r > \mathrm{R}_\delta$.

(¹) The square bracket here denotes the integral part of $r^{\rho_1(r)}$.

Generalising a previous calculation § (III, 14) we have, for all points off the negative real axis,

$$\log P(z) = \int_0^\infty \frac{z n(x)}{x(x+z)} dx.$$

We assume that $-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}$, ($\varphi = \arg z$), and we divide up the integral into four parts, I_1, I_2, I_3, I_4 with ranges of integration $0, R_\delta$; $R_\delta, r/\lambda$; $r/\lambda, r\lambda$; $r\lambda, +\infty$, where R_δ is the number defined above and λ is a number greater than 1. Observing that $|x+z|$ is greater than x and r we obtain, as in § (III, 7).

$$|I_1| < K_\delta, \qquad |I_2| < \frac{\lambda^{\delta-\rho_1}}{\rho_1 - \delta} r^{\rho_1(r)}, \qquad |I_4| < \frac{\lambda^{\rho_1+\delta-1}}{1-\rho_1-\delta} r^{\rho_1(r)};$$

and, writing

$$I(a, b) = \int_a^b \frac{z\, x^{\rho_1(r)}}{x(x+z)} dx,$$

$$|I(0, R_\delta)| < K_\delta, \ |I(R_\delta, r/\lambda)| < \frac{\lambda^{\delta-\rho_1}}{\rho_1-\delta} r^{\rho_1(r)}, \ |I(\lambda r, \infty)| < \frac{\lambda^{\rho_1+\delta-1}}{1-\rho_1-\delta} r^{\rho_1(r)}.$$

It follows moreover from the properties of $\rho_1(r)$ that in every segment $r/\lambda, r\lambda$

$$n(x) = (1 + \eta(x))\, x^{\rho_1(r)} \qquad (|\eta(x)| < \varepsilon)$$

for any given value of $\lambda > 1$ and for sufficiently large values of r $(r > r_{\lambda, \varepsilon})$. Hence

$$I_3 = I(r/\lambda, r\lambda) + \varepsilon(r) \int_{r/\lambda}^{r\lambda} x^{\rho_1(r)-1} dx.$$

The last integral is less than $\lambda^{\rho_1-\delta} r^{\rho_1(r)}/(\rho_1 - \delta)$. We may therefore write

$$\log P(z) = I(0, \infty) + \theta r^{\rho_1(r)} [K_1 \lambda^{\delta-\rho_1} + K_2 \lambda^{\rho_1+\delta-1} + \varepsilon(r)\lambda^{\rho_1-\delta}], \quad (|\theta| < 2).$$

The first two terms in the square bracket can be made as small as we please by choice of λ and the third tends to zero as r tends to infinity. That is to say the term in the brackets tends to zero as r tends to infinity.

It remains to evaluate the integral $J(z) = I(0, \infty)$. This presents no difficulty. We replace x by the complex variable $x = te^{i\psi}$ and observe that, if for example $\varphi > 0$, the integral

$$\int_C \frac{x^{\rho_1(r)} z}{x(x+z)} dx$$

taken along the arc $x = |t|$, $0 \leqslant \psi \leqslant \varphi$, tends to zero as t tends to infinity, since $|x(x+z)| > t^2$. We now make the change of variable $x = te^{i\varphi}$, giving

$$J(z) = e^{i\varphi\rho_1(r)} J(r),$$

and putting $t = ur$,

$$J(r) = r^{\rho_1(r)} \int_0^\infty \frac{u^{\rho_1(r)}}{1+u} du = \frac{\pi}{\sin[\pi\rho_1(r)]} r^{\rho_1(r)}.$$

Finally, since $\rho_1(r)$ tends to ρ_1, we have in the region under consideration

$$(5, 3) \quad V_1(z) = \log P(z) \backsim \frac{\pi}{\sin(\pi\rho_1)} e^{i\varphi\rho_1} r^{\rho_1(r)}, \qquad \left(-\frac{\pi}{2} \leqslant \varphi \leqslant \frac{\pi}{2}\right).$$

If in this function $V_1(z)$ we now replace z by $\sqrt{z}$ we obtain a new function $V(z)$ regular in the region $r > R_1$, $-\pi \leqslant \varphi \leqslant \pi$ and such that

$$V(z) \backsim \frac{\pi}{\sin(\pi\rho_1)} r^{\rho_1(\sqrt{r})/2} e^{i\varphi\rho_1/2}.$$

And this function satisfies the conditions imposed if

$$\frac{\pi}{\sin(\pi\rho_1)} r^{\rho_1(\sqrt{r})/2} = r^{\rho(r)}, \qquad \rho_1 = 2\rho.$$

The reader will have no difficulty in seeing that if $\rho(r)$ is a proximate order L, the function $\rho_1(r)$ defined by the equation

$$\rho_1(x) = 2\rho(x^2) + 2\frac{\log\sin(2\pi\rho) - \log\pi}{\log x}$$

is also a proximate order L.

Theorem 36 is thus completely established.

A number of deductions concerning integral functions of finite order greater than $1/2$ have been made from this theorem of Wiman. We shall content ourselves with mentioning only one, as follows :

THEOREM 37. — *If $f(z)$ is an integral function of finite order ρ and q the smallest integer greater than 2ρ, there is a positive number h and a sequence of circles of indefinitely increasing radius on each of which there are arcs of total measure not less than $2\pi/q$ on which*

$$\log |f(z)| > h \log M(r).$$

Consider the function

$$G(z^q) = [f(z) - a][f(z\omega) - a] \ldots [f(z\omega^{q-1}) - a],$$

where ω is a primitive q'th root of unity and a a constant less than 1. $G(u)$ is clearly an integral function of order less than $1/2$. If $\rho(r)$ is a general proximate order of $f(z)$, $U = |u|$, and $G_1(U)$ is the maximum modulus of $G(u)$, we have plainly

$$\log G_1(U) < (q+1)r^{\rho(r)} = (q+1)U^{\rho_1(U)} \qquad \left(\rho_1(U) = \frac{1}{q}\rho(U^{\frac{1}{q}})\right).$$

Here as in the former case $\rho_1(U)$ is also a proximate order. On the other hand if $n(x, a)$ is the number of zeros of the function $f(z) - a$, and consequently of all the other functions in the product also, we have by Jensen's theorem

$$\log G_1(U) > (q-1)\int_3^r n(x, a)dx/x.$$

But, in virtue of theorem 25, we can choose an a such that the right-hand side of this inequality shall be greater than $Hr^{\rho(r)}$ for a sequence of indefinitely increasing values of r, and therefore greater than $HU^{\rho_1(U)}$ for a sequence of indefinitely increasing values of U. From these two inequalities we conclude that

$$\overline{\lim_{U=\infty}} \frac{\log G_1(U)}{U^{\rho_1(U)}} = K > 0,$$

where K lies between H and $(q+1)$. Therefore the function

$$\rho_1(U) + \frac{\log K}{\log U}$$

is a proximate order L of $G(u)$. And by Wiman's theorem we have, on a sequence of circles,

$$|G(u)| = |f(z) - a| \ldots |f(z\omega^{q-1}) - a| > e^{hU^{\rho_1(U)}} = e^{hr^{\rho(r)}}$$
$$\geqslant [M(r)]^h \qquad (h > 0).$$

This inequality is clearly equivalent to the enunciation of the theorem.

3. Functions of zero order. — An integral function equal to 1 at the origin and of zero order is of the form

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{a_n}\right),$$

with exponent of convergence zero. The logarithm of the maximum modulus is less than

$$\log \prod_1^\infty \left(1 + \frac{r}{r_n}\right) = \int_0^\infty \frac{r\,n(x)}{x(x+r)}\,dx < \int_0^r \frac{n(x)}{x}\,dx + r\int_r^\infty \frac{n(x)}{x^2}\,dx,$$

and comparing this inequality with Jensen's theorem we see that

$$(5,4)\quad \log M(r) = \int_0^r \frac{n(x)}{x}\,dx + \theta r\int_r^\infty \frac{n(x)}{x^2}\,dx \qquad (0 < \theta < 1).$$

Now $\log M(r)$ increases more rapidly than $K \log r$. Therefore those functions for which the coefficient of θ is less than $K \log r$ have the following remarkable property : the logarithm of the maximum modulus is asymptotically equivalent to the integral of theorem (16), or

$$(5,5)\qquad \log M(r) \backsim \int_0^r n(x)\,dx/x.$$

In order that the condition may be fulfilled it is necessary that $n(r)$, the number obtained on substituting $n(r)$ for $n(x)$ in the coefficient of 0, should be less than $K \log r$. And it is easy to see that this condition is also sufficient. Now if $n(r) < K \log r$ it follows from (5, 5) that $\log M(r) < K (\log r)^2$; and conversely, as is shewn by Jensen's theorem. Thus all functions subject to the condition

$$(5,\ 6) \qquad \overline{\lim_{r=\infty}} \frac{\log M(r)}{(\log r)^2} < +\infty.$$

satisfy equation (5, 5). Such functions bear a close resemblance to polynomials, but we shall see that in fact the analogy goes even further than this.

We proceed to examine the modulus of the function outside certain small regions enclosing the zeros.

Let $\beta(x)$ be a decreasing function tending to zero but greater than $1/x$, and let us consider the sequence of circles of centre $z = 0$ and radii

$$R_1, \quad R_2 = (1 + \beta(R_1))R_1, \quad \ldots\ldots, \quad R_{m+1} = (1 + \beta(R_m))R_m, \quad \ldots .$$

The numbers R_m form an indefinitely increasing sequence. Let D_m be the annulus between the two circles of radii R_m and R_{m+1}, and D'_m that between the circles of radii $R'_m = R_m(1 - \beta(R_m))$ and $R''_m = R_m(1 + 2\beta(R_m))$, m_1 and m_2 being the values of $n(r)$ for $r = R'_m$ and R''_m respectively. In Boutroux's theorem concerning polynomials we have a weapon for dealing with the modulus of the function in the annulus D'_m. Consider the polynomial

$$g_m(z) = \prod_{m_1+1}^{m_2} \left(1 - \frac{z}{a_n}\right),$$

when z lies in the annulus D'_m. We have

$$|g_m(z)| > \prod_{m_1+1}^{m_2} \left|\frac{r - r_n}{R''_m}\right|,$$

and so, writing

$$r = R'_m + 3x\beta R_m, \qquad r_n = R'_m + 3\alpha_n\beta R_m,$$

$$|g_m(z)| > \left[3\frac{\beta}{1+2\beta}\right]^{m_2-m_1}\left|\prod_{m_1+1}^{m_2}(x-\alpha_n)\right| \qquad (\beta = \beta(R_m)).$$

In this last polynomial both x and α_n are real and lie between 0 and 1. Therefore, except on certain circles whose radii lie in a set of intervals of total length $\frac{3}{H}\beta(R_m)R_m$, we have

$$|g_m(z)| \geqslant \left(\frac{2e}{\beta}\right)^{-Hm_2}.$$

Now H is at our disposal and we may put $\frac{3}{H} = (\beta(R_m))^2$. The inequality

$$(5, 7) \qquad \log|g_m(z)| \geqslant -Kn(R''_m)(\beta(R_m))^{-2}$$

is then satisfied throughout D_m except in certain annuli whose total width is less than $R_m\beta^2$.

A similar procedure shews that the same inequality holds good except in certain sectors in which the variation of $\varphi = \arg z$ does not exceed $K\beta^2$. It is in fact sufficient to observe that

$$\left|1-\frac{z}{a_n}\right| > \sin|\varphi-\varphi_n| \qquad (\varphi_n = \arg a_n).$$

Finally, if the inequality (5, 7) holds on the boundary of a domain containing no zeros of $g_m(z)$ it holds throughout the domain. We see therefore that this inequality is satisfied at all points of D_m save in certain domains which can be enclosed in curvilinear trapeziums, of dimensions less than $R_m\beta(R_m)^2$, surrounding the zeros. The region Δ_m of points of D_m exterior to these trapeziums is a connected region and repeating the argument for the annuli D_{m-1}, D_{m+1} it is clear that together with the two analogous regions in D_{m-1} and D_{m+1}, it forms another connected region.

If z is a point of Δ_m we have

$$|f(z)| \geqslant \prod_{1}^{m_1} \left(1 - \frac{r_n}{r}\right) \prod_{m_2+1}^{\infty} \left(1 - \frac{r}{r_n}\right) e^{U(r)}$$

$$U(r) = \int_0^r n(x)\, dx/x - K m_2 \beta^{-2}.$$

The logarithm of the first product is greater than

$$m_1 \log \beta > -m_2/\beta$$

since $r_n < r(1 - \beta)$.

In the second product $r_n > r(1 + 2\beta)/(1 + \beta)$, and so

$$\log\left(1 - \frac{r}{r_n}\right) > -\frac{1 + 2\beta}{\beta} \frac{r}{r_n}.$$

But, by § (III, 2),

$$\sum_{m_2+1}^{\infty} \frac{1}{r_n} < \int_r^{\infty} n(x)\, dx/x^2.$$

Hence

$$\log \prod_{m_2+1}^{\infty} \left(1 - \frac{r}{r_n}\right) > -\frac{K}{\beta} r \int_r^{\infty} n(x)\, dx/x^2.$$

Finally, since

$$\int_r^{\infty} n(x)\, dx/x^2 > \int_{R''_m}^{\infty} n(x)\, dx/x^2 > n(R''_m)/R''_m,$$

we obtain the inequality

$$(5,\ 8). \qquad \log |f(z)| > \int_0^r \frac{n(x)}{x}\, dx - \frac{K}{\beta^2} r \int_r^{\infty} \frac{n(x)}{x^2}\, dx$$

valid in the domain Δ formed by adding together all the regions Δ_m.
In particular for functions satisfying condition (5, 6) the coefficient

of K/β^2 is less than $K \log r$ and we may choose $\beta(x)$ decreasing so slowly that the last term on the right in (5, 8) will be trivial. Hence the following theorem :

THEOREM 38. — *For all functions satisfying condition* (5, 6) *the logarithm of the modulus is asymptotically equivalent to the logarithm of the maximum modulus corresponding to the same value of r, except in certain small domains enclosing the zeros, the dimensions of such domains being infinitesimal in comparison with their distance from the origin.*

A striking consequence of this theorem is that, for all $r > r_a$, all the zeros of the function $f(z) - a$ lie in the excluded domains. The problem of the distribution of the zeros of functions of zero order of the class we have considered may thus be regarded as solved to a first approximation.

It is easy to shew that *for functions $f(z)$ of zero order which do not satisfy condition* (5, 6) *the logarithm of the minimum modulus of $f(z)$ is still asymptotically equivalent to the logarithm of the maximum modulus on an infinite sequence of circles of indefinitely increasing radii.*

In virtue of (5, 4) and (5, 8) it is sufficient to prove that the ratio

$$\int_0^r \frac{n(x)}{x}\,dx \Big/ r \int_r^\infty \frac{n(x)}{x^2}\,dx$$

is unbounded. For if the ratio is very large for a certain value of r it will be very large in the segment $r, 2r$, since the integral in the denominator is a decreasing function of r. If the ratio were bounded we could find a number h such that ultimately

$$U(r) = \int_0^r \frac{n(x)}{x}\,dx < hr \int_r^\infty \frac{n(x)}{x^2}\,dx = hr \int_r^\infty U'(x)\,dx/x,$$

and, integrating by parts,

$$U(r) < -hU(r) + hr \int_r^\infty U(x)\,dx/x^2,$$

which we may write

$$hV(r) + (1 + h)rV'(r) > 0,$$

$V(r)$ denoting the last integral written above. On integration it follows that $V(r)r^{h/(h+1)}$ is an increasing function, and consequently

$$\int_r^\infty U(x)\,dx/x^2 > K\,r^{-h/(h+1)}.$$

But this inequality implies that $U(r)r^{-1/(h+1)}$ does not tend to o, so that, since $\log M(r)$ is greater than $U(r)$, the order of the function must be at least equal to $1/(1+h)$, which is contrary to our hypothesis. The result stated, which is due to Littlewood, is therefore proved.

II. — Paths of determination of functions of finite order.

Let C be a path of finite determination a for the function $F(z)$. That is to say that if $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots$ is a decreasing sequence of numbers tending to zero, there is a point of C corresponding to each ε_n beyond which

$$|F(z) - a| < \varepsilon_n.$$

There is then a domain D_n in which $|F(z) - a| < \varepsilon_n$, while on the contour $|F(z) - a| = \varepsilon_n$, and from a certain point onwards the curve C is interior to D_n. Inside a circle $|z| < r$, D_n may consist of several portions and it is clear that each of these portions is simply connected. Any curve which is ultimately interior to every domain D_n corresponding to the sequence is a path of determination a. *We shall say that the various paths of determination a obtained in this way are contiguous.* Between these paths $F(z)$ tends to a. The set of contiguous paths of determination a constitutes what Boutroux has call a "langue", which we propose to call a *tract of determination a*. Tracts may differ in two respects, either because they correspond to different asymptotic values or because the paths of determination taken in two tracts are not contiguous.

It follows from the property of continuity that given a path of determination a we can find a contiguous path of the same determination consisting of rectilinear segments. Such a curve cuts a circle $|z| = r$ in a finite number of points only — a property which will be of use to us later.

4. Some results due to Lindelof. — Lindelöf has proved by a simple application of theorem 33 that *if the function* F(z) *is bounded in a domain* D *limited by two paths of determination* a, *then in this domain* F(z) *tends uniformly to* a.

It will be convenient in this case to assume that the essential singularity is at the origin instead of at infinity. We suppose further that a preliminary transformation (z, z^3) has been made such that the arcs of the circle $|z| = r$ intercepted by every domain D_r, where D_r is that part of D lying in the circle $|z| = r$ $(r < R)$ and having the origin as a frontier point, are interior to an arc of angular measure less than $2\pi/3$. The function F(z) transforms into a function $F_1(z)$ regular on two paths C and C′ (we may regard these paths as consisting of rectilinear segments) and in the open domain D between them and interior to a circle $|z| < A$. Finally we assume that the asymptotic value a is zero, as we may clearly do without diminishing the generality of the proposition, and take K to be the bound of $F_1(z)$ in the domain D.

Consider a point z_0 interior to a circle $|z| \leqslant r < A$ and let z_0' be the inverse point with respect to the circle. The points z_0, z_0' are unaffected by the linear transformation

$$\frac{Z - z_0}{Z - z_0'} = - \frac{z - z_0}{z - z_0'}.$$

A point of the circumférence is transformed into a point of the circumference and the interior of the circle is consequently transformed into itself. As z_0 tends to zero the transformation becomes a rotation through the angle π. We can therefore find a number $k < 1$ such that, if $|z_0| < kr$, an arc of the circumference of length $2\pi r/3$ is transformed into another non-overlapping arc.

Now let z_0 be a point of D_r of modulus less than kr and make this transformation. The domain D_r is transformed into a domain D_r' and those parts of the boundaries of D_r and D_r' which are arcs of the circle $|z_0| = r$ have no common point. The domain D_r'' common to D_r, D_r' is bounded only by arcs of C, C′ and their transformations, and z_0 is an interior point of D_r''. The function $F_1(z)$ transforms into F (Z) and, since z_0 is invariant, $F_2(z_0)$ is equal to $F_1(z_0)$. If ε_r is the

maximum modulus of $F_1(z)$ on those portions of C and C′ bounding D_r, then

$$|F_1(z)\,F_2(z)| < K\varepsilon_r$$

at all points of the boundary of D_r'' except at the origin and its transformation if these points are on the boundary. But $|F_1(z)\,F_2(z)|$ does not exceed K^2 in the domain D_r'', and so we can apply theorem 33. Therefore

$$|F_1(z_0)|^2 < K\varepsilon_r.$$

Now this inequality is satisfied at every point z_0 interior to D_r and such that $|z_0| < kr$ and we see that $F_1(z)$ tends uniformly to zero as z approaches the origin by a path interior to D or along the boundary. Lindelöf's result is therefore proved. From it we can deduce the following fundamental theorem :

THEOREM 39. — *If the function* F(z) *is bounded in a domain* D *lying between two paths of finite determination, then the asymptotic values on these paths are equal and* F(z) *tends uniformly to their common value along any receding path in* D.

To prove this we suppose the theorem false. The asymptotic values of $F(z)$ along the two paths C and C′ are then distinct, say a and b, $a \neq b$. The function

$$\left[F(z) - \frac{a+b}{2}\right]^2$$

tends to $(a-b)^2/4$ on the two paths C and C′ and is bounded in the domain D between them. It therefore tends uniformly to this value in the domain D and on the curves C, C′. But this implies that $F(z)$ will tend uniformly either to a or to b in D and on C and C′, which is contrary to our hypothesis. The theorem is thus established, for we know that when the asymptotic values on C and C′ are equal and the function is bounded in the domain D between them it tends uniformly to their common value along any receding path in D.

Corollary 39. — *Two different tracts of finite determination are separated by at least one path of determination ∞.*

For if we consider two paths on which the asymptotic values are different F(z) is unbounded in the two domains between them and the same argument as was used in § 1 to establish the existence of a path of determination ∞ in general suffices to shew that there is such a path in each of the two domains.

If the asymptotic values for the two tracts are the same, F(z) cannot be bounded in either of the two domains between them, for if it were the two tracts would reduce to one, by theorem 39.

It would be possible to give numerous examples of properties of paths of determination proved by means of theorem 33. Suppose for example *that* F(z) *has a path of determination with an asymptote, which we may suppose to be parallel to the positive real axis. Then if* ρ *is the order of* F(z) *and* $\beta > 0$ *there is no other path of finite determination not contiguous to the first in any angle of magnitude* $\pi/(\rho + \beta)$ *containing the positive real axis.*

In fact we consider the function

$$\omega(z) = e^{kz^{\rho'}} \qquad (\rho < \rho' < \rho + \beta)$$

the number k, of modulus 1, being chosen so that in the angle in question $\log|\omega(z)| > Kr^{\rho'}$. This is certainly possible if we are given the choice of ρ'. Then theorem 33, with this function $\omega(z)$, shews that if F(z) is bounded on two paths in the angle it is bounded in the domain between them. The two paths must therefore be contiguous and the proposition is proved.

Results of this class are easily generalised by means of conformal representations.

By way of illustration let us consider the function

$$f(z) = \int_0^z [\sin(z^{m/2})]^2 z^{-m} dz,$$

where m is an integer greater than 1. Since the derivative of this function $f(z)$ is plainly of order $m/2$, the function $f(z)$ itself is also of this order. Putting $z = re^{2i\pi p/m}$ (p an integer) we have

$$f(z) = e^{2i\pi p/m} \int_0^r [\sin r^{m/2}]^2 r^{-m} dr.$$

As r tends to infinity this integral tends to a positive limit. So $f(z)$ tends to different limits in the m directions of argument $2\pi p/m$ and there are at least m tracts corresponding to different asymptotic values. In a similar manner we find that the directions of argument $(2p+1)\pi/m$ are paths of determination ∞. And so it follows from the remark above, since the angle between a path of finite determination and the two nearest paths of determination ∞ is $\pi/m < \pi/\rho$, that there is no path of finite determination between these paths which is not contiguous to the path of finite determination considered. *The function $f(z)$ of order $\rho = m/2$ has therefore 2ρ finite tracts.*

The function $\sin(\sqrt{z})/\sqrt{z}$ similarly furnishes an example of a function of order $1/2$ with one finite tract.

5. The number of tracts of a function of finite order. — In this paragraph we prove by a method due to Carleman that the number of finite tracts of a function $F(z)$ of finite order ρ does not exceed 5ρ.

Consider two receding paths C and C′ on which $F(z)$ is bounded, while it is unbounded in D, one of the domains between them (We might suppose that C and C′ coincide and that D consists of the plane $|z| > R$ cut along C). We assume as usual that C and C′ have been deformed so that they only cut a circle $|z| = r > R$ in a finite number of points. The number of arcs of $|z| = r$ intercepted by the domain D is therefore finite. Let the total length of these arcs be denoted by $r = \theta(r)$ and let $e^{V(r)}$ denote the maximum modulus of $F(z)$ on these arcs, regarded as closed. $V(r)$ is by hypothesis unbounded and is therefore ultimately, by Cauchy's theorem, an increasing function of r. Thus, beyond a certain value of r, the maximum value of $V(r)$ is attained at interior points of these arcs and is greater than the upper bound of $F(z)$ on C and C′. Further, the curve of maximum modulus in D has the same properties as the curve of maximum modulus for the whole plane and the function $V(r)$ is differentiable except at isolated points. In what follows we shall denote by D a domain between the two paths C and C′ and exterior to a circle $|z| = R$ of sufficiently large radius to ensure that all the above conditions are satisfied for $|z| = r > R$.

We observe that if we make a transformation $z = Z^{\mu}$, (μ an integer)

and if D′ is one of the connected domains derived from D by this transformation, then the transformation of $F(z)$ is regular in D′ and on the finite part of its contour. $V(r)$ is transformed into $V(R^\mu)$ and $\theta(r)$ in $\theta(R^\mu)/\mu$.

Returning to the domain D and denoting by e^k the upper bound of $|F(z)|$ on C, C′ and those arcs of the circle $|z| = R$ which are parts of the contour of D we have, by hypothesis, for $r > R$,

$$\overset{\frown}{V}(r) > k.$$

Let r and r' be numbers such that $R < r' < r$ and z_0 a point of modulus r' for which $\log |F(z_0)|$ is equal to $V(r')$. z_0 is interior to D. Let D_r be that part of D which is interior to the circle $|z| \leqslant r$. The maximum of $|F(z)|$ in D_r is attained on the circle $|z| = r$ and is equal to $e^{V(r)}$, and on that part of contour of D_r interior to $|z| \leqslant r$ we have $|F(z)| \leqslant e^k$.

It is easy to construct a function regular at interior points of the circle $|z| \leqslant r$ and on the open arcs of its circumference both interior and exterior to D_r, such that the modulus of the function is equal to $e^{V(r)}$ on the interior arcs and e^k on the exterior arcs. For if $re^{i\alpha}$ and $re^{i\beta}$ are two points of the circumference, these two are the only critical points of the function

$$U(z, \alpha, \beta) = e^{h'}[(z - re^{i\alpha})/(z - re^{i\beta})]^{hi}, \qquad [U(o, \alpha, \beta) = e^{h' - h(\beta - \alpha)}]$$

of constant modulus on the two arcs defined by the two points. Moreover $|U(z, \alpha, \beta)|$ and its reciprocal are bounded in the circle. Taking

$$h = (V(r) - k)/\pi, \qquad h' = (\alpha - \beta)h/2$$

the logarithm of $|U|$ will be equal to $V(r) - k$ on one of the arcs and zero on the other. We can construct a product of such functions and, multiplying by e^k, we obtain a function $U(z)$ regular in the circle. Both $|U(z)|$ and $|1/U(z)|$ are bounded in the circle and $|U(z)|$ is equal to $e^{V(r)}$ on the arcs interior to D_r and to e^k on the exterior arcs. Applying corollary 33 to the function $1/U(z)$ we see that in the circle $|U(z)|$ is greater than or equal e^k.

The function

$$F(z)/U(z)$$

is regular in $D_{r'}$ and on the contour except at the points of intersection of $|z|=r$ with C and C', and its modulus is less than or equal to 1 on the contour except at these points and is bounded in $D_{r'}$. Therefore by corollary 33 the modulus does not exceed 1 in $D_{r'}$, and we have*

$$V(r') \leqslant \log |U(z_0)|.$$

Now, from the mode of construction of $U(z)$, we have

$$\log |U(z_0)| = k + \frac{1}{\pi}(V(r) - k)\theta(r, r'),$$

where

$$\theta(r, r') = \theta_1(r, r') - \frac{1}{2}\theta(r),$$

$\theta_1(r, r')$ denoting the sum of the angles subtended at z_0 by the arcs of the circle $|z|=r$ intercepted by D. $\theta(r, r')$ is a maximum when there is only one such arc of length $\theta(r)$ bisected by the radius through z_0, and an elementary geometrical calculation shews that

$$\theta(r, r') \leqslant 2 \operatorname{arc} \tan \left[\frac{r+r'}{r-r'} \tan \left(\frac{1}{4}\theta(r)\right)\right].$$

Thus

$$V(r') - k \leqslant \frac{2}{\pi}(V(r) - k) \operatorname{arc} \tan \left(\frac{r+r'}{r-r'} \tan \left(\frac{1}{4}\theta(r)\right)\right)$$

or

$$\frac{V(r) - V(r')}{(r-r')(V(r')-k)} \geqslant \frac{1}{(r-r')}\left[\frac{\pi}{2 \operatorname{arc} \tan \left(\frac{r+r'}{r-r'} \tan \frac{\theta(r)}{4}\right)} - 1\right].$$

Suppose that r is a point at which $V(x)$ has a derivative and let $r' \to r$.* Then

$$r\frac{V'(r)}{V(r)-k} \geqslant \frac{1}{\pi} \cot \frac{\theta(r)}{4}.$$

Had we made the transformation indicated at the outset we should have the same inequality with r replaced by R^{μ} and $\theta(r)$ by $\theta(R^{\mu})/\mu$ and the left-hand side multiplied by μ. We have then, for all values of μ

$$r\frac{V'(r)}{V(r)-k} \geqslant \frac{1}{\pi\mu}\cot\frac{\theta(r)}{4\mu}$$

and, since the left-hand side is independent of μ, we may replace the expression on the right by its limit when $\mu = \infty$. Thus

$$\frac{V'(r)}{V(r)-k} \geqslant \frac{4}{\pi}\frac{1}{r\theta(r)}.$$

Integrating from R to r, and bearing in mind that $V(r)$ is an indefinitely increasing function, we obtain Carleman's fundamental inequality,

(5, 9) $$\log V(r) + K \geqslant \frac{4}{\pi}\int_R^r \frac{dx}{x\theta(x)},$$

K being a finite number.

In particular $\log M(r) \geqslant V(r)$ and must therefore satisfy this inequality, where $\theta(x)$ may be replaced by its maximum value 2π. Hence, when $F(z)$ is bounded on two receding paths and is unbounded in the intervening domain the inequality

(5, 10) $$\varliminf_{r=\infty} \frac{\log V(r)}{\log r} \geqslant \frac{2}{\pi^2} > \frac{1}{5}$$

must be satisfied; and in particular, *in order that there should exist a receding path on which* $F(z)$ *is bounded the condition*

(5, 11) $$\varliminf_{r=\infty} \frac{\log_2 M(r)}{\log r} \geqslant \frac{2}{\pi^2}$$

is necessary.

It will be observed that the numbers occurring in these two inequalities are not the order, but what we shall call the *lower order* (left-hand side of (5, 11)) and *the lower order between* C *and* C′ (left-hand side of (5, 10)). A receding path on which $F(z)$ is bounded will be said to be a *path of finite indetermination.*

The result may now be stated as follows :

If there is to be a path of finite indetermination the lower order must be at least equal to $2/\pi^2$.

Now suppose that there are p non-contiguous paths of finite indetermination. That is to say that $F(z)$ is unbounded in the intervening domains. Applying formula (5, 9) to these p intervening domains and adding we obtain the inequality

$$\log M(r) + K \geqslant \frac{4}{\pi p}\int_R^r \left(\sum \frac{1}{\theta_j(x)}\right)\frac{dx}{x},$$

since each function $V(r)$ is less than or equal to $\log M(r)$. The integrand is the sum of the reciprocals of p numbers of sum at most 2π and is therefore at least equal to $p\frac{p}{2\pi}$. And so, substituting this value we have

$$\varliminf_{r=\infty} \frac{\log_2 M(r)}{\log r} \geqslant \frac{2p}{\pi^2} > \frac{p}{5}.$$

The following proposition is now established :

THEOREM 40. — *That a function* $F(z)$ *should have* p *non-contiguous paths of finite indetermination it is necessary that its lower order should be at least to* $2p/\pi^2$. *In particular a function of order* ρ *has at most* 5ρ *distinct tracts and* 5ρ *distinct finite asymptotic values.*

In the same way we can shew that for a function to have p non-contiguous paths of finite indetermination in an infinite domain it is necessary that the lower order in this domain should be at least $2(p-1)/\pi^2$.

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§ 5. Carleman **1**.

CHAPTER VI

Generalisations of Picard's theorem.

Picard obtained his original result concerning the values which a uniform function assumes in the neighbourhood of an isolated essential singularity by the use of the elliptic modular function. We have seen in chapters III and IV that a direct investigation of the moduli of the zeros of such a function leads to a result more complete than that discovered by Picard in the first instance. The first of these investigations was due to Borel, and later, Landau and Schottky, using the methods introduced by him, obtained generalisations of another kind. It has since been shewn by Caratheodory and Lindelöf that, from this last point of view, the modular function is essential to the proof of the most complete results. In short the analytical weapon used by Picard is no mere technical convenience, but of fundamental importance in the theory.

In this chapter we shall be principally concerned with functions regular in a certain circle in which they omit to assume two exceptional values, say 0 and 1. We shall then pass, by means of simple transformations, to the consideration of more general regions and applications to the theory of the functions $F(z)$. Schottky and Landau have shewn that a function $\Phi(z)$ regular in the circle $|z|<r$, in which it does not assume the values 0 and 1, has its modulus in this circle less than

$$e^{K\frac{r}{r-|z|}}.$$

There are numerous deductions to be made from this. For instance it follows immediately that Picard's theorem applies to a function regular in a circle in which its modulus is of sufficiently high order. A simple transformation by inversion leads to complementary results about the functions $F(z)$.

As a further consequence of Schottky's theorem we have the fact that any set of functions regular in a certain circle in which they omit to assume the values 0 and 1 constitutes what Montel has called a *normal family*. That is to say that from every sequence of these functions we can select another sequence converging uniformly to a limiting function, which may be constant or infinite. Following Montel we are able to deduce from this certain properties of functions which are regular in an angle in which they do not assume the values 0 and 1. From an investigation of the properties of the special family $F(z\sigma^n)$ Julia has recently deduced a number of very remarkable results in this department of the subject. Some of the more important of these are discussed in the final section.

I. — The Schottky-Landau theorem and its direct applications.

1. The properties of the modular function. — It is known that the Weierstrassian elliptic function $p(u)$, of periods $2\omega_1$ and $2\omega_2$, satisfies the differential equation

$$p'^2 = 4p^3 - g_2 p - g_3, \tag{6, 1}$$

where g_2 and g_3 are functions of ω_1 and ω_2, and the ratio g_2^3/g_3^2 depends only on the ratio of the periods $\omega = \omega_1/\omega_2$. We shall assume that the imaginary part of the complex number ω is positive, or $I(\omega) > 0$. This can always be brought about by changing the sign of one of the periods. The ratio g_2^3/g_3^2 is unchanged by any substitution of the arithmetic modular group; that is to say if ω is replaced by $\omega' = (a\omega + b)/(c\omega + d)$, where a, b, c, d are integers such that $ad - bc = 1$.

The modular function

$$J(\omega) = \frac{g_2^3}{g_2^3 - 27 g_3^2}$$

is thus also unchanged by any substitution of this group, and in particular

$$J\left(-\frac{1}{\omega}\right) = J(\omega).$$

This function can be easily expressed in terms of the variable

$$q = e^{2i\pi\omega}.$$

We have in fact

(6, 2) $$y = J(\omega) = \left[\frac{1}{12} + 20\sum_{1}^{\infty} n^3 \frac{q^n}{1-q^n}\right]^3 \Big/ q \prod_{1}^{\infty} (1-q^n)^{24}.$$

This expression shews that $J(\omega)$ is regular in the half-plane $[I(\omega) > 0$, corresponding to $0 < |q| < 1]$.

It can be shewn that this function assumes every value once and once only in the fundamental region of the modular group

$$\left(-\frac{1}{2} < \Re(\omega) \leqslant \frac{1}{2}, \quad |\omega| > 1, \quad \text{and } \omega = 1, \quad 0 \leqslant \Re(\omega) \leqslant \frac{1}{2}\right).$$

This property enables us to invert the elliptic integral.

The inverse function $\omega(y)$ of $J(\omega)$ is a multiform function with an infinity of branches. The only singularities of this function are the points $0, 1, \infty$; and $I(\omega(y))$ is always positive.

Making the transformation

$$x = \frac{4}{27} \frac{(1-y+y^2)^3}{y^2(y-1)^3}$$

the function $\nu(x) = \omega(y)$ becomes equal to the ratio of the two periods of the elliptic integral

(6, 3) $$\int dz / \sqrt{z(z-1)(z-x)}.$$

The singularities of $\nu(x)$ are also the points $0, 1, \infty$ and $I(\nu)$ is always positive.

These properties of $J(\omega)$ and its inverse form the basis of our subsequent work and will be regarded as known.

The expression of $J(\omega)$ as a function of q shews that the point $q=0$ is a simple pole and that the product $qJ(\omega)$ is regular for $|q| < 1$. Now

$$|q| = e^{-2\pi I(\omega)};$$

and so, if K is the maximum modulus of $q\mathrm{J}(\omega)$ for $|q| = e^{-2\pi}$, we have

$$\text{(a)} \qquad |\mathrm{J}(\omega)| \leqslant \mathrm{K}e^{2\pi \mathrm{I}(\omega)}, \qquad (\mathrm{I}(\omega) \geqslant 1).$$

It is easy to deduce from this a similar inequality for the modulus of $\mathrm{J}(\omega)$ when $\mathrm{I}(\omega) < 1$. In fact equation (6, 2) shews that the coefficients in the Laurent series for $\mathrm{J}(\omega)$ expressed as a function of q, valid in the circle $|q| < 1$, are real and positive. $\mathrm{J}(\omega)$ is therefore dominated by the function obtained on replacing q by $|q|$; that is to say by the function $\mathrm{J}(i\mathrm{I}(\omega))$. Hence

$$|\mathrm{J}(\omega)| \leqslant \mathrm{J}(i\mathrm{I}(\omega)) = \mathrm{J}(-1/i\mathrm{I}(\omega)) = \mathrm{J}(i/\mathrm{I}(\omega)).$$

Now if $0 < \mathrm{I}(\omega) < 1$, then $1/\mathrm{I}(\omega)$ is greater than 1 and applying inequality (a), we obtain the complementary inequality

$$\text{(b)} \qquad |\mathrm{J}(\omega)| \leqslant \mathrm{K}e^{2\pi/\mathrm{I}(\omega)} \qquad (0 < \mathrm{I}(\omega) < 1).$$

So, comparing (a) and (b), we have, for all ω, $(\mathrm{I}(\omega) > 0)$,

$$(6,\ 4) \qquad |\mathrm{J}(\omega)| < \mathrm{K}e^{2\pi\left[\mathrm{I}(\omega) + \frac{1}{\mathrm{I}(\omega)}\right]}.$$

K has a determinate numerical value. *This inequality is fundamental in Landau's proof.*

We shall also have need of the following function-theory proposition known as Schwarz's Lemma. It is an immediate consequence of theorem I.

Let the function $\Phi(z)$ be regular in the circle $|z| < r$ and zero at the origin. Suppose further that $|\Phi(z)|$ is less than or equal to a number M *at all interior points of this circle.* Now if z_0 is an interior point of the circle we can find ρ such that $|z_0| < \rho < r$. The function $\Phi(z)/z$ is regular in the circle $|z| \leqslant \rho$ and on its circumference $|\Phi(z)/z|$ is less than M/ρ. Therefore, by Cauchy's theorem,

$$|\Phi(z_0)/z_0| < \mathrm{M}/\rho.$$

But the function on the left is independent of ρ and is therefore

less than or equal to the limit of the right-hand side for $\rho = r$. Thus, for all interior points of the circle $|z| < r$, we have

$$|\Phi(z)| \leqslant M\frac{|z|}{r}. \tag{6, 5}$$

This is Schwarz's inequality.

2. Picard's proof. — Before proceeding to the proof of Landau's inequality we will shew how Picard's theorem can be deduced from a single property of the function $\omega(y)$ (or $\nu(x)$) — the property that a branch $\varpi(y)$ of the function is capable of analytic continuation along any curve so long as the point y does not pass through any of the points $0, 1, \infty$; and the imaginary part is of constant sign.

Consider an integral function $f(z)$ which does not assume the values 0 and 1. Let

$$\varpi(y) = A_0 + A_1(y - f(0)) + \ldots\ldots$$

be the expansion in series of a branch of the function $\omega(y)$ in the neighbourhood of the point $y = f(0)$. If in this series we replace y by the function $f(z)$ we obtain a function

$$g(z) = \varpi(f(z)) = A_0 + A_1 f'(0)z + \ldots\ldots$$

regular in the neighbourhood of the origin. Now this function $g(z)$ is an integral function. For, as z recedes along a radius $\varphi =$ constant, $y = f(z)$ does not assume the values $0, 1, \infty$; $\varpi(y)$ can be continued analytically and so we encounter no singularity of $g(z)$. Moreover the imaginary part of $g(z)$ is always positive and this, as we saw in § (II, 1) implies that $g(z)$ is a constant. Therefore $f(z)$ is also constant, and Picard's original theorem is proved viz :

An integral function which omits to assume two exceptional values reduces to a constant.

3. Landau's inequality. — Consider a function

$$\Phi(z) = c_0 + c_1 z + \ldots$$

regular in the circle $|z|<r$, in which it omits to assume the values 0 and 1. Let

$$\varpi(y) = A_0 + A_1(y - c_0) + \ldots..$$

be the expansion of a branch of $\omega(y)$ in the neighbourhood of c_0. The preceding argument shews once more that the function

$$g(z) = \varpi(\Phi(z)) = A_0 + A_1 c_1 z + \ldots..$$

is regular in the circle $|z|<r$. On the other hand we know that the imaginary part of $g(z)$, and therefore of A_0, is positive. Now write

$$h(z) = \frac{g(z) - A_0}{g(z) - A_0'} = \frac{g_1(z) - a_0 + i[g_2(z) - b_0]}{g_1(z) - a_0 + i[g_2(z) + b_0]}$$

where $g_1(z)$, $g_2(z)$ are the real and imaginary parts of $g(z)$, a_0 and b_0 the real and imaginary parts of A_0, and A_0' the conjugate of A_0. Now $g_2(z) + b_0$ is positive, and so $h(z)$ is regular in the circle $|z|<r$, since the denominator does not vanish. We see further that $|h(z)|<1$ and that $h(0) = 0$. Hence, writing

$$h(z) = ue^{i\psi},$$

we have throughout the circle, by Schwarz's lemma,

$$|h(z)| = u \leqslant \frac{|z|}{|r|}.$$

Now

$$g(z) = \frac{A_0' h(z) - A_0}{h(z) - 1} = a_0 + ib_0 \frac{1 + h(z)}{1 - h(z)},$$

and, equating imaginary parts, this leads to

$$\begin{aligned} I[g(z)] = g_2(z) &= b_0 \Re\left[\frac{1 + h(z)}{1 - h(z)}\right] = b_0 \Re\left[\frac{1 + ue^{i\psi}}{1 - ue^{i\psi}}\right] \\ &= b_0 \frac{1 - u^2}{1 + u^2 - 2u\cos\psi}. \end{aligned}$$

The coefficient of b_0 is clearly a minimum when $\cos\psi = -1$.

and a maximum when $\cos\psi = 1$. Its value thus lies between $\dfrac{1-u}{1+u}$ and $\dfrac{1+u}{1-u}$; or, since $u<1$, between $\dfrac{1}{2}(1-u)$ and $2/(1-u)$. Thus, in virtue of the inequality for u, we have

$$b_0 \frac{r-|z|}{2r} < \mathrm{I}[g(z)] < b_0 \frac{2r}{r-|z|}.$$

But $\Phi(z)$ is equal to $\mathrm{J}(g(z))$, since $\mathrm{J}(\omega)$ is the inverse of $\omega(y)$. Therefore, by (6, 4),

$$\log|\Phi(z)| = \log|\mathrm{J}[g(z)]| < 2\pi\left[\mathrm{I}(g(z)) + \frac{1}{\mathrm{I}(g(z))}\right] + \log \mathrm{K}.$$

Replacing $\mathrm{I}(g(z))$ and $[\mathrm{I}(g(z))]^{-1}$ by their upper bounds we obtain the inequality

$$\log|\Phi(z)| < 4\pi\left(b_0 + \frac{1}{b_0}\right)\frac{r}{r-|z|} + \log \mathrm{K}.$$

A fortiori, since $r > r - |z|$,

$$\log|\Phi(z)| < \chi(\Phi(0))\frac{r}{r-|z|}, \tag{6, 6}$$

where

$$\chi(x) = 4\pi\left[\mathrm{I}(\omega(x)) + \frac{1}{\mathrm{I}(\omega(x))}\right] + \log \mathrm{K}. \tag{6, 7}$$

In fact we have the following theorem due to Landau :

THEOREM 41. — *If the function $\Phi(z)$ is regular in the circle $|z|<r$ in which it omits to assume the values* 0 *and* 1, *then its modulus satisfies the inequality* (6, 6) *at all interior points of the circle.*

There is an immediate deduction from this theorem concerning functions which are regular and of order greater than unity in a given circle. We must first define what we mean by the order of such a function. We shall say that *a function $\Phi(z)$, regular in the unit*

circle $|z| < 1$, *is of order* ρ *in this circle when its maximum modulus* $M(r)$, $(r < 1)$, *satisfies the equation*

$$(6, 8) \qquad \overline{\lim_{r=1}} \frac{\log_2 M(r)}{-\log(1-r)} = \rho.$$

We shall shew that Picard's theorem holds for these functions of order greater than unity also. The proposition may be precisely stated as follows :

Any function $\Phi(z)$ *regular in the unit circle and whose maximum modulus satisfies the condition*

$$(6, 9) \qquad \overline{\lim_{r=1}} (1 - r) \log M(r) = +\infty$$

assumes every value, with one possible exception, an infinite number of times in the circle.

For let us suppose that there are two exceptional values 0 and 1. We can then draw a circle $|z| = \delta$ such that the annulus $\delta \leqslant |z| < 1$ is free from zeros of the functions $\Phi(z)$ and $\Phi(z) - 1$. Consider a circle Γ, $|z| = \delta + \frac{1}{2}(1 - \delta)$. $\Phi(z)$ is continuous on the circumference of this circle and does not assume the values 0 and 1. So $\omega[\Phi(z)]$ is continuous and its imaginary part has a determinate maximum and minimum. The function $\chi[\Phi(z)]$ has therefore a finite upper bound h on the circumference of Γ. Now let z_0 be any point of the circumference of Γ. The circle of centre z_0 and radius $\frac{1}{2}(1 - \delta)$ is free from zeros of $\Phi(z)$ and $\Phi(z) - 1$, and so we have

$$\log|\Phi(z)| < h \frac{1 - \delta}{1 - \delta - 2|z - z_0|}.$$

In particular we see that at all points on the circle $|z| = r' > \delta + \frac{1 - \delta}{2}$

$$\log|\Phi(z)| < h \frac{1 - \delta}{2(1 - r')}.$$

and

$$(1 - r') \log M(r') < \frac{1}{2} h(1 - \delta),$$

which contradicts our hypothesis (6, 9).

The theorem is therefore proved.

The example

$$\Phi(z) = e^{\frac{1}{1-z}}$$

serves to shew that where condition (6, 9) is not fulfilled there may actually be an infinity of exceptional values. Writing $z = re^{i\theta}$, we have

$$\log |\Phi(z)| = (1 - r\cos\theta)/(1 + r^2 - 2r\cos\theta),$$

and, for a given value of r, this varies between $1/(1+r)$ and $1/(1-r)$; so that for all values of r between 0 and 1 it varies between $1/2$ and $+\infty$, and the function $\Phi(z)$ does not assume any value of modulus less than $\sqrt{e}$.

4. Applications to the functions of the class F(z). — We consider a function $F(z)$ having an essential singularity at the origin and we assume that the order ρ is greater than $1/2$. It is proposed to make a study of the values of this function in an angle subtended at the origin; that is to say in a domain in which the argument of z is confined between two given numbers, while the modulus r is sufficiently small to ensure that the function shall be regular in the domain. Let Δ denote the angle $\alpha < \varphi < \beta$, ($\varphi = \arg z$), and $M(r, \Delta)$ the maximum modulus of $F(z)$ on the arc of the circle $|z| = r$ intercepted by Δ. Then the order of $F(z)$ in Δ is defined to be

$$\overline{\lim_{r=0}} \frac{\log_2 M(r, \Delta)}{-\log r}.$$

Since $F(z)$ is of order ρ at the essential singularity at the origin there is clearly at least one angle of given magnitude $\gamma < 2\pi$ in which $F(z)$ is of order ρ.

Given ε such that $\frac{\pi}{\rho} + \varepsilon < 2\pi$ let Λ be a small angle in which

$F(z)$ is of order ρ and let Δ be an angle containing Λ and of magnitude $\frac{\pi}{\rho}+\varepsilon$. It can be shewn that $F(z)$ *assumes every value, with one possible exception, an infinity of times in the angle* Δ.

We may suppose that the positive real axis bisects the angle Δ internally. The substitution $z=Z^\beta$, $\pi\beta=\frac{\pi}{\rho}+\varepsilon$, where Z^β is real on the positive real axis, transforms the angle Δ into the half-plane Δ' to the right of the imaginary axis and $F(z)$ into a function $\Phi(Z)$ regular in this half-plane (for $|Z|<R_0$). $\Phi(Z)$ is *a fortiori* regular in a circle Γ in this half-plane with the imaginary axis as its tangent through the origin. We may suppose that this circle is of radius unity. $\Phi(Z)$ is regular in Δ' and its only singularity on the circumference is at the origin. Now the angle Λ has been transformed into an angle Λ' in the half-plane Δ' and in this angle $\Phi(Z)$ is of order $\rho\beta>1$. If P is a point of Λ' interior to Γ the ratio of its distances from the origin and the circumference remains between two fixed numbers, K and $1/K$, provided both these distances are sufficiently small. Now let Z' be a point of the arc $|Z|=R$ intercepted by Λ', at which $\Phi(Z)$ attains its maximum $M(R, \Lambda')$. The distance from this point to the circumference of Γ is at least equal to R/K. Therefore if we draw a circle of radius $1-\frac{R}{K}$ concentric with Γ the maximum modulus of $\Phi(Z)$ in this circle, which we shall denote by $M\left(1-\frac{R}{K}, \Gamma\right)$, is not less than $M(R, \Lambda')$; or

$$M\left(1-\frac{R}{K}, \Gamma\right) \geqslant M(R, \Lambda').$$

Hence

$$\overline{\lim_{x=1}}\, \frac{\log_2 M(x, \Gamma)}{-\log(1-x)} \geqslant \overline{\lim_{R=0}}\, \frac{\log_2 M(R, \Lambda')}{-\log R} \geqslant \rho\beta > 1.$$

$\Phi(Z)$ is thus of order greater than 1 in the circle Γ and therefore assumes every value, with one possible exception, in this circle. *A fortiori* the same is true of $F(z)$ in the angle Δ, and we have the following result :

THEOREM 42. — *A function* $F(z)$ *of order* $\rho>1/2$ *assumes every value, with one possible exception, an infinity of times in any angle of*

magnitude greater than π/ρ_1 *in the interior of which the function is of order* ρ_1, $(1/2 < \rho_1 \leqslant \rho)$.

The statement that a function $F(z)$ is of order ρ_1 in the interior of a certain angle is defined to mean that this angle contains another in which $F(z)$ is of order ρ_1.

In particular in the case of functions of infinite order there is always at least one angle of arbitrarily small magnitude in which the fonction assumes every value, with only one possible exception. In general these angles are to be found by searching for those directions in the plane in which the function is effectively of infinite order. Thus a function of infinite order with positive real coefficients assumes every value, save possibly one exceptional value, in any angle containing the positive real axis.

A simple deduction from the results of chapter V is the following: *a function of order* $\rho > 1/2$ *is of order* ρ *in the interior of any angle of magnitude greater than* $\pi\left(2 - \frac{1}{\rho}\right)$

Suppose the proposition false. Then in the domain exterior to the angle Δ defined by

$$-\pi < \varphi(2\rho + \alpha) < \pi \qquad (\alpha > 0),$$

and on the bounding radii, $F(z)$ is of order $\rho_1 < \rho$. So the product

$$F(z)e^{-z^{\rho'}} \qquad (\rho_1 < \rho' < \rho)$$

tends to zero on the bounding radii of the angle Δ and is of order ρ inside it. Therefore, by Phragmèn's theorem 34, this product is bounded in Δ and it follows that $F(z)$ is of order less than ρ in the whole plane, which is contrary to hypothesis.

This last result, together with that of theorem 42, enables us to prove a result due to Bieberbach.

Consider first the case of a function $F(z)$ of order $\rho \geqslant 1$. In every angle of magnitude $2\pi - \frac{\pi}{\rho} + \varepsilon$, $F(z)$ is of order ρ, and there is an angle A interior to this one in which $F(z)$ is of order ρ. Now theorem 42 holds in any angle of magnitude $\frac{\pi}{\rho} + \varepsilon$ containing A. But,

since $\frac{\pi}{\rho}+\varepsilon$ is not greater than $2\pi-\frac{\pi}{\rho}+\varepsilon$, this latter angle satisfies the necessary condition, so that in it $F(z)$ assumes every value, with one possible exception, an infinity of times.

If ρ lies between $1/2$ and 1 (or is equal to 1) every angle of magnitude $\frac{\pi}{\rho}+\varepsilon$, greater than or equal to $2\pi-\frac{\pi}{\rho}+\varepsilon$, is an angle in which $F(z)$ is of order ρ and, by theorem 42, $F(z)$ assumes every value, with not more than one exception, an infinity of times in such an angle. These two results together constitute Bierberbach's theorem :

THEOREM 43. — *A function $F(z)$ of order ρ, greater than $1/2$, assumes every value, with one possible exception, an infinity of times in any angle of magnitude greater than $2\pi-\frac{\pi}{\rho}$ if $\rho \geqslant 1$, and in any angle of magnitude greater than $\frac{\pi}{\rho}$ if $\rho \leqslant 1$.*

It is clear that in this case, as in that of theorem 42, the exceptional value is not necessarily the same for every angle in the plane.

II. — NORMAL FAMILIES OF REGULAR FUNCTIONS.

5. Families of regular functions. — Montel has obtained interesting results concerning sets of functions defined in a simply or multiply connected domain D bounded by one or more contours, which we shall suppose to be made up of arcs of simple curves (in applications we shall only have to deal with straight lines and circular arcs). The theory of such sets of functions is to be our next topic.

A set or *family of functions*, regular in such a domain D, is said to be *bounded in the aggregate in the domain* D when in every closed region D′, completely interior to D, the modulus of every function of the family is less than a fixed number $M_{D'}$.

THEOREM 44. — *From every sequence of regular functions bounded*

in the aggregate in a domain D *it is possible to select a sequence of functions uniformly convergent in* D.

A sequence of functions will be said to be *uniformly convergent in a domain* D *when it is uniformly convergent in every closed region interior to* D. By theorem 5 its limit function is regular in D.

We first prove the theorem in the particular case in which the domain D is a circle.

Let

$$\Phi_1(z),\ \Phi_2(z),\ \ldots\ldots,\ \Phi_n(z),\ \ldots\ldots$$

be a sequence of functions regular and bounded in the aggregate in a circle $|z| \leqslant r$ and on the circumference. Let M be the upper bound of the modulus. We have

$$\Phi_n(z) = c_0{}^n + c_1{}^n z + \ldots + c_p{}^n z^p + \ldots$$

and, by Cauchy's inequality (1, 4),

(6, 9) $$|c_p{}^n| < M/r^p$$

for all values of n and p. Hence, for all values of n, the modulus of the remainder

$$R_p{}^n(z) = c^n{}_{p+1} z^{p+1} + \ldots\ldots$$

of $\Phi_n(z)$ is less than

(6, 10) $$\left(\frac{\rho}{r}\right)^p \frac{Mr}{r-\rho}$$

in the circle $|z| \leqslant \rho < r$.

Now the sequence $c_0{}^1,\ c_0{}^2,\ \ldots,\ c_0{}^n,\ \ldots$ has one or more limiting points. Let $c_0{}'$ be such a limit. Clearly, by (6, 9),

$$|c_0{}'| \leqslant M$$

and there is a sequence of numbers

$$c_0{}^{n_0^1},\quad c_0{}^{n_0^2},\quad \ldots\ldots,\quad c_0{}^{n_0{}^q},\quad \ldots\ldots$$

selected from the sequence $\{c_0{}^n\}$, tending to $c_0{}'$. If $c_1{}'$ is a limit of the sequence $\{c_1{}^{n_0{}^q}\}$ $(q = 1, 2, 3, \ldots)$, then, by (6, 9),

$$|c_1{}'| \leqslant M/r$$

and there will be a sequence $\{c_1^{n_1^q}\}$, selected from $\{c_1^{n_0^q}\}$, with c_1' as its sole limit. Proceeding in this way, if the sequence $\{c_p^{n_p^q}\}$ converges to c_p' we select from the sequence $\{c_{p+1}^{n_p^q}\}$ a subsequence $\{c_{p+1}^{n^q_{p+1}}\}$ converging to a limit c'_{p+1}, whose modulus does not exceed M/r^{p+1}. We thus obtain a sequence

$$c_0', \quad c_1', \quad \ldots\ldots, \quad c'_p, \quad \ldots\ldots$$

such that

$$|c'_p| \leqslant M r^{-p} \text{ (1)}$$

The function

$$\Phi(z) = \sum_0^\infty c'_p z^p$$

will be regular in the circle $|z| < r$ and the modulus of its remainder $R_p(z)$ will not exceed the expression (6, 10). We assert that the sequence of functions.

$$\Phi_{n_1^1}(z), \quad \Phi_{n_2^2}(z), \quad \ldots\ldots, \quad \Phi_{n_q^q}(z), \quad \ldots\ldots$$

converges to $\Phi(z)$ uniformly in every circle $|z| \leqslant \rho < r$. By construction the first P coefficients ($P < q$) of $\Phi_{n_q^q}(z)$ belong to the sequences whose respective limits are the first P numbers c_p' and their ranks in these sequences are at least $q - P$. Therefore, P being fixed, the difference

$$[\Phi(z) - R_P(z)] - \left[\Phi_{n_q^q}(z) - R_P^{n_q^q}(z)\right]$$

tends uniformly to zero in the circle $|z| \leqslant \rho$ as q tends to infinity. Moreover, since the two remainders $|R_P(z)|$ and $|R_P^{n_q^q}(z)|$ are neither of them greater than the expression (6, 10), we can choose P so that they are as small as we please for all values of n_q^q and all points in the circle $|z| \leqslant \rho$. Uniform convergence in this circle is thus established.

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(1) Certain or all of the numbers c'_p may be zero. As for example, if

$$\Phi_n(z) = 1/n(1 - z).$$

Now consider a finite domain D and a region D_1 interior to D. Every point of the contour of D_1 is the centre of a circle lying entirely inside D. Similarly for every interior point of D_1. We can therefore find (Heine-Borel theorem) a finite set of circles $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$, all interior to D and such that every point of D_1 is also an interior point of one of them. Moreover every circle Γ_q may be regarded as interior to a concentric circle Γ_q', itself interior to D. The result we have just proved holds good in the circle Γ_q'. We can thus select from the sequence of functions $\Phi_n(z)$ a partial sequence S_1 uniformly convergent in the closed circle Γ_1. From S_1 we can select another partial sequence S_2 uniformly convergent in the closed circle Γ_2. It is clear that S_2 is uniformly convergent in both the circles Γ_1 and Γ_2. In this way we obtain after p operations a sequence S_p uniformly convergent in all the circles Γ_q. and so in the region covered by them. S_p is therefore uniformly convergent in D_1.

Now let us consider a sequence of regions $D_1, D_2, \ldots, D_p, \ldots$ all interior to D and each interior to its successor in the sequence and such that D_p tends to D. We have seen that there is a partial sequence S_1', selected from a given sequence, uniformly convergent in D_1. From this sequence we can select another S_2' uniformly convergent in D_2 and so on. The sequence obtained by taking the first function in S_1', then the second in S_2', and so on, will be convergent in the whole domain D; that is to say in every region D' interior to D. For we can choose p so that D' shall be interior to D_p.

Montel's theorem is thus completely established.

6. Normal families of regular functions. — *A family of functions regular in a domain* D *is said to be normal when from every sequence of functions of the family it is possible to select a partial sequence converging uniformly in the domain* D *to a limiting function, or to a constant,* which may be infinite.

A family is said to be *normal at an interior point of* D when this point is the centre of a circle, lying entirely inside D, in which the family is normal.

A family which is normal in a domain is plainly normal at every point of the domain. Conversely, as may be proved by precisely the same argument as was used in the second part of the proof of

Montel's theorem, a family which is normal at every point of a domain is normal in the domain.

It follows that if a family of functions is not normal in a certain domain, then that domain contains at least one point at which the family is not normal. That is to say there is a point P such that in every circle of centre P lying in the domain at least one sequence of functions of the family is not normal.

By theorem 44 a family of functions regular and bounded in the aggregate in a domain D is normal in this domain. It is clear that in this case the limit of a convergent sequence of functions of the family is either a regular function or a finite constant.

THEOREM 45. — *If the functions of a family normal in a domain* D *are uniformly bounded at an interior point of* D (that is to say that at a point z_0 of D the moduli of these functions are less than a fixed number $M(z_0)$), *then the functions of the family are bounded in the aggregate in* D.

In the first place the limit of a convergent sequence of functions of the family is either a function regular in D or, since the value of the limiting function is finite at z_0, a finite constant. Suppose that in a region D′ interior to D the functions of the family are not bounded in the aggregate; that is to say that there is a sequence of functions $\Phi_n(z)$, belonging to the family, such that the maximum modulus of $\Phi_n(z)$ in D′ tends to infinity with n. From this sequence we can select another converging uniformly to a limit in a region D″ containing D′ in its interior. The limiting function of this partial sequence cannot be infinity and is therefore bounded on the contour of D′. The maximum modulus of the functions of the sequence convergent on the contour of D′ must therefore also be bounded. But this is contrary to our hypothesis, and so proves the theorem.

7. Families of functions omitting to assume the values 0 and 1 in a domain. — Consider the set of functions regular in the circle $|z| < 1$ and omitting to assume two exceptional values, say 0 and 1, in this circle. *This set of functions is a normal family.* To prove this let us consider a sequence $\Phi_1(z)$, $\Phi_2(z)$, ..., $\Phi_n(z)$, ... of func-

tions of the family. There are now two possibilities. Either the sequence $\{\Phi_n(0)\}$ has at least one limit different from 0, 1 and ∞, or it has not. Consider the first case, and let a be the limit in question. There is a sequence $\{\Phi_{n_q}(z)\}$ selected from $\{\Phi_n(z)\}$ such that $\{\Phi_{n_q}(0)\}$ tends to a. The sequence of numbers $\chi(\Phi_{n_q}(0))$ figuring in Landau's inequality will be bounded, since their limit is $\chi(a)$. The functions $\Phi_{n_q}(z)$ will therefore be bounded in the aggregate in the circle $|z| < 1$, and, for $|z| \leqslant r < 1$, we have

$$\log |\Phi_{n_q}(z)| < K/(1 - r).$$

Let us now turn to the second case. The sequence of functions $\{\Phi_n(z)\}$ is then such that the only possible limiting values of the sequence $\{\Phi_n(0)\}$ are 0, 1 and ∞. Suppose that 0 is a limiting value and let $\{\Phi_{n_q}(z)\}$ be a partial sequence such that $\{\Phi_{n_q}(0)\}$ tends to 0. The sequence of functions

$$\Psi_{n_q}(z) = \frac{1}{2}[1 + \sqrt{\Phi_{n_q}(z)}],$$

where the particular determination of the square-root is arbitrary, is again a sequence of functions regular in the unit circle, since $\Phi_{n_q}(z)$ does not vanish. These functions $\Psi_{n_q}(z)$ omit the values 0 and 1 in the circle, for they can only be equal to 0 or to 1 if $\Phi_{n_q}(z)$ assumes the value 1. Moreover the sequence $\{\Psi_{n_q}(0)\}$ tends to the limit $1/2$. So, by the preceding argument, the functions $\Psi_{n_q}(z)$ are bounded in the aggregate in the circle. But

$$\Phi_{n_q}(z) = [1 - 2\Psi_{n_q}(z)]^2$$

and the functions $\Phi_{n_q}(z)$ are therefore also bounded in the aggregate in the unit circle.

If the limit of the sequence $\Phi_{n_q}(0)$ is 1 instead of 0, the same argument applies, with $\Phi_{n_q}(z) - 1$ in place of $\Phi_{n_q}(z)$.

Thus in the three cases so far considered it is possible to select from the sequence $\{\Phi_n(z)\}$ another sequence of functions bounded in the aggregate and therefore normal.

It remains to examine the case where the sole limiting value of the sequence $\{\Phi_n(0)\}$ is ∞.

Consider the sequence of functions

$$\Psi_n(z) = 1/\Phi_n(z),$$

regular in the circle, since $\Phi_n(z)$ does not vanish, and omitting to assume the values 0 and 1. The sequence $\{\Psi_n(0)\}$ tends to zero. The functions $\Psi_n(z)$ are therefore bounded in the aggregate in the circle and we can select a sequence of functions $\{\Psi_{n_q}(z)\}$ converging uniformly in the circle to a limiting function $\Psi(z)$. This function $\Psi(z)$ is identically zero. For suppose that it is not. The equation

$$\Phi_{n_q}(z) = 1/\Psi_{n_q}(z)$$

shews that the sequence of functions $\Phi_{n_q}(z)$ also converges uniformly on the circumference of a circle $|z| = r < 1$ on which there are no zeros of $\Psi(z)$. Hence, by Weierstrass' theorem 5, this sequence converges uniformly in this circle $|z| < r$ and its limit is a function regular in the circle. Therefore $\{\Phi_{n_q}(0)\}$ cannot tend to ∞. But this is contrary to our hypothesis. It follows therefore that $\Psi(z)$ is zero, the sequence $\Phi_{n_q}(z)$ converges uniformly to infinity and the sequence $\Phi_n(z)$ is normal.

Thus every sequence of functions $\Phi_n(z)$ regular and different from 0 and 1 in the unit circle has the following property : it is possible to select from it a sequence uniformly convergent in the circle. The family of functions regular and omitting to assume the values 0 and 1 in the unit circle is in fact *normal*.

Since a family normal at all points of a domain is normal in the domain this result can be extended to a general domain D of the form considered above. We have then :

THEOREM 46. — *Every family of functions regular and omitting to assume two exceptional values in a connected domain* D *is a normal family in the domain.*

One important observation is suggested by the foregoing proof. We saw that if we consider a sequence of functions $\{\Phi_n(z)\}$ belonging to the family and if the values $\Phi_n(z_0)$ of these functions at the point z_0 tend to infinity, then every convergent sequence selected from the sequence $\{\Phi_n(z)\}$ has infinity as its limit. We are able to prove the following general proposition.

Corollary 46. — *If, given a sequence of functions $\Phi_n(z)$ normal in a domain D, there exists a region D′ interior to D in which the sole limit, as n tends to infinity, of the maximum modulus of $\Phi_n(z)$, is infinity, then the sequence of functions $\Phi_n(z)$ converges uniformly to infinity throughout the domain* D.

In the first place, at every point z_0 of D′ the sequence of numbers $\Phi_n(z_0)$ has the point at infinity as its sole limiting point. For if there were another limit we could select from the sequence $\{\Phi_n(z)\}$ a partial sequence $\{\Phi_{n_q}(z)\}$ of functions uniformly bounded at z_0 and, this sequence being normal, the functions $\Phi_{n_q}(z)$ would be bounded in the aggregate in D (theorem 45) and their maximum modulus would be bounded on the contour of D′, which is contrary to hypothesis. The numbers $\Phi_n(z_0)$ therefore tend to infinity and it follows that every convergent sequence selected from the sequence $\{\Phi_n(z)\}$ converges uniformly to infinity. We can assert that in any region D″ interior to D the minimum modulus of the functions $\Phi_{n_q}(z)$ of any sequence selected from $\{\Phi_n(z)\}$ is unbounded. For as we have just seen, we can select from the sequence $\{\Phi_{n_q}(z)\}$ a partial sequence converging uniformly to infinity in D, and therefore in D″. Therefore the sequence $\{\Phi_n(z)\}$ itself converges to infinity uniformly in D. This proves the theorem.

8. Application to the functions F(z). — Let us consider a function $\Phi(z)$ regular in the interior of a sector Δ defined by the inequalities

$$0 < \varphi < \beta \leqslant 2\pi, \qquad 0 < r < R_0,$$

where r and φ denote the modulus and argument of z, and suppose that $\Phi(z)$ omits to assume the values 0 and 1 in this sector. It is proposed to investigate the behaviour of $\Phi(z)$ as z approaches the origin by certain paths in the sector.

Let R be a number less than R_0, ε a number less than $\beta/2$ and D_1 the closed region defined by the inequalities

$$\varepsilon \leqslant \varphi \leqslant \beta - \varepsilon, \qquad \frac{1}{2}R \leqslant r \leqslant R.$$

There is always a domain D_1' interior to Δ and containing D_1. Let z be a point of the domain D_1'. Then the functions of the sequence

$$\Phi(z), \quad \Phi\left(\frac{z}{2}\right), \quad \ldots\ldots, \quad \Phi\left(\frac{z}{2^n}\right), \quad \ldots\ldots$$

are regular in the domain D_1' just mentioned, $\Phi_n(z) = \Phi\left(\frac{z}{2^n}\right)$ being the value of $\Phi(z)$ when z lies in the domain D'_n deduced from D_1' by the transformation $\left(z, \frac{z}{2^n}\right)$, and a fortiori in the region D_n defined by

$$\varepsilon \leqslant \varphi \leqslant \beta - \varepsilon, \qquad \frac{R}{2^{n+1}} \leqslant r \leqslant \frac{R}{2^n},$$

which are both parts of the sector Δ. The study of $\Phi(z)$ in the neighbourhood of the origin is thus reduced to that of the sequence $\Phi_n(z)$ in the closed region D_1. Now since the function $\Phi(z)$ omits the values o and 1 in Δ the sequence $\{\Phi_n(z)\}$ is normal in the domain D_1' containing D_1.

Let OL be a straight line, $\varphi =$ constant, in Δ. We can choose ε so that the regions D_n of the sequence intercept OL. Let d denote the interval on OL interior to D_1'.

Now suppose that $\Phi(z)$ remains bounded as z approaches the origin along OL. The functions $\Phi_n(z)$ are then uniformly bounded on d, whence it follows by theorem 45 that they are bounded in the aggregate in D_1'. Their modulus is thus less than some fixed number at every point of D_1. Therefore, since ε is arbitrary, *$\Phi(z)$ is bounded in every sector interior to Δ.*

Precisely the same argument, where we consider $1/\Phi(z)$, or $1/[\Phi(z) - 1]$ suffices to shew that *if $\Phi(z)$ or $\Phi(z) - 1$ is always greater than some fixed positive number on* OL, *then this property holds* (with another positive number) *in every sector interior to Δ.*

Suppose that on OL $\Phi(z)$ tends to a finite limit a. The functions $\Phi_n(z)$ tend to a uniformly in the interval d. Therefore a is the limit of every convergent partial sequence selected from the sequence $\Phi_n(z)$. It follows that *the functions $\Phi_n(z)$ converge uniformly to a throughout the region* D_1. For suppose this assertion false. We can then find a partial sequence $\{\Phi_{n_q}(z)\}$, selected from this sequence

of functions $\Phi_n(z)$, such that the maximum modulus of $(\Phi_{n_q}(z) - a)$ on the contour of the region D_1 remains greater than a fixed number. Now from this sequence $\{\Phi_{n_q}(z)\}$ we can select a partial sequence $\{\Phi_{n'_q}(z)\}$ converging to a uniformly in D, and therefore on the contour of D_1, which is contrary to hypothesis. *We thus see that the sequence of functions $\Phi_n(z)$ converges to a uniformly as z approaches the origin by a path in any sector interior to Δ.*

In virtue of corollary 46, if $|\Phi(z)|$ tends to ∞ as z approaches the origin by a path interior to a certain sector inside Δ, then this property holds uniformly in any sector inside Δ. The same result is true if $\Phi(z)$ tends to either of the exceptional values 0 or 1.

These results, due in part to Lindelöf and in part to Montel, are collected in the following theorem :

THEOREM 47. — *$F(z)$ is a function with an isolated essential singularity at infinity and omits to assume the values a and b, $a \neq b$, in an angle Δ, $\alpha < \varphi < \beta$, subtended at the origin. We then have the following results.*

(i) *If $F(z)$ is bounded on a straight line, $\varphi =$ constant, interior to Δ, then it is bounded in every angle interior to Δ; if an angle interior to Δ contains a path of determination ∞, then $F(z)$ tends to infinity uniformly in every angle interior to Δ. If $F(z) - a$ or $F(z) - b$ is greater than a fixed number on a straight line or tends to zero along some path, then these properties hold uniformly in any angle interior to Δ.*

(ii) *If $F(z)$ tends to a limit along a receding rectilinear path, $\varphi =$ constant, inside Δ, then it tends to this limit uniformly in every angle interior to Δ*

This proposition enables us in certain cases to find angular domains in which $F(z)$ assumes every value, with only one possible exception, an infinity of times.

Suppose for example that $F(z)$ is bounded on a rectilinear path OL, which we may regard as coinciding with the positive real axis. Then we assert that *there is an angle subtended at the origin, of arbitrarily small magnitude ε, in which $F(z)$ assumes every value, with one possible exception, an infinity of times.*

Suppose the contrary to be the case. Then in every angle of magnitude ε there are at least two exceptional values which $F(z)$ does not assume for $r > R_\Delta$. Choose n such that $n\varepsilon$ is greater than 4π and divide the plane up into n equal sectors with the origin as their common vertex. Let $OL_1, OL_2, \ldots, OL_n$, be the dividing radii of the sectors and let OL_1 coincide with the positive real axis OX. The radii are numbered in the order in which we meet them in making a circuit round the origin in the positive sense. Let Δ_p be the domain bounded by OL_p and OL_{p+1} and outside the circle $|z| > R'$. The number n has been chosen so that the domain formed by two adjacent sectors lies inside a sector of magnitude less than ε. Now consider the domain formed by the junction of the two sectors Δ_n and Δ_1. This domain is interior to an angle of magnitude less than ε, which is therefore one in which $F(z)$ omits to assume two exceptional values, provided $|z| > R'$ (R' being sufficiently large). Since $F(z)$ is bounded on OL_1 it is bounded throughout Δ_1, Δ_n and on OL_2 and OL_n. Applying the same argument to Δ_1 and Δ_2 we see that $F(z)$ is bounded in Δ_2 and on OL_3, and so on. $F(z)$ is thus bounded in each sector and on its bounding radii, and therefore in the whole plane exterior to the circle $|z| > R'$. But this is impossible. Our proposition that there is at least one angle of magnitude ε in which $F(z)$ assumes every value, save possibly one exceptional value, an infinity of times, is therefore proved.

III. — Some theorems due to Julia.

9. The family of functions $F(z\sigma^n)$. — Let the function $F(z)$ be regular in the domain exterior to a circle $|z| > R_0$ except at infinity where it has an essential singularity, and let σ be a number of modulus greater than 1. We proceed to consider the sequence of functions

$$F_n(z) = F(z\sigma^n).$$

They are all regular in the domain exterior to a circle $|z| > R_0$ (ex-

cept at infinity) and the values of the functions $F_n(z)$ in the annular region C, defined by

$$R_1 \leqslant |z| \leqslant R_1|\sigma| \qquad (R_1 > R_0),$$

are equal to the values of $F(z)$ in the annular regions C_n, defined by

$$R_1|\sigma|^n \leqslant |z| \leqslant R_1|\sigma|^{n+1}.$$

These annuli together cover the whole plane exterior to the circle $|z| = R_1$ and we shall seek, by investigating the sequence of functions $F_n(z)$ in C, to obtain results concerning the values of $F(z)$ in the whole plane. The functions $F_n(z)$ are regular in an annular domain C′ containing C. The maximum modulus of $F_n(z)$ in C tends to infinity with n, since it is equal to the maximum modulus of $F(z)$ in the circle $|z| \leqslant R_1|\sigma|^{n+1}$. It follows that if the family $F_n(z)$ is a normal family, then the functions $F_n(z)$ tend uniformly to infinity in the closed annular region C (Corollary 46).

But the functions $F_n(z)$ cannot tend uniformly to infinity. For if this were so the modulus of $F(z)$ would always exceed a given number outside a certain circle $|z| \geqslant r$. But this stands in contradiction with Weierstrass theorem (theorem 4).

The family of functions $F(z)$ *is therefore not normal in* C′. — There is thus at least one point P in this annulus at which the family is not normal. But C′ is any annular domain containing C, and it follows that there is at least one point z_0 of C, (z_0 may be on one of the bounding circles) at which the sequence $F_n(z)$ is not normal.

Let Γ be a circle of centre z_0 and arbitrarily small radius d. In this circle the family $F_n(z)$ is not normal. Now suppose that there are two values a and b which are assumed only a finite number of times in the circle by the sequence of functions $F_n(z)$. Then from every sequence selected from $\{F_n(z)\}$ we can select another sequence of functions which do not assume the values a and b. The family $F_n(z)$ will then be normal, by Montel's theorem 46. But this is a contradiction and it follows that the functions $F_n(z)$ assume every value in the circle, save possibly a single exceptional value a_{z_0}.

It follows that if Γ′ is the circle of centre $z_0\sigma^n$ and radius $d|\sigma|^n$

the function $F(z)$ assumes every value, with only one possible exception a_{z_0}, an infinity of times in the sequence of circles Γ_n.

We may therefore state the following proposition in completion of Picard's theorem :

THEOREM 48. — *If the function* $F(z)$ *has an isolated essential singularity at infinity and if* σ *is a number of modulus greater than* 1, *there is at least one point* z_0 *with the following property : d being an arbitrary positive number,* $F(z)$ *assumes every value, save possibly a single exceptional value* a_{z_0}, *an infinity of times in the sequence of circles* Γ_n *described about the centres* $z_0\sigma^n$ *with radii* $d|\sigma|^n$.

The exceptional value, if it exists, is independent of the radius of the circle Γ. There are, in fact, two possibilities : either there is no exceptional value, however small d may be; or else, for a certain value of d, there is an exceptional value a_{z_0}. In the second case we may replace Γ by a circle of radius less than d and it is clear that in this circle the functions $F_n(z)$ still assume the value a_{z_0} a finite number of times only. This is therefore still the exceptional value. It is possible however that there may be an exceptional value for $d < \delta$ and none for $d > \delta$.

Let a be a number different from the exceptional value a_{z_0}, if it exists, and let $d_1, d_2, \ldots, d_p, \ldots$ be a sequence of numbers tending to zero, and S_p the sequence of circles $\Gamma_p{}^n$ of radii $d_p|\sigma|^n$ and centres $z_0\sigma^n$. Whatever value p may have the function $F(z) - a$ has an infinity of zeros in the sequence of circles S_p. Denote by Γ^1 a circle of the sequence S_1 containing a zero $z_1{}^a$ of $F(z) - a$, and by Γ^2 a circle of S_2 containing another zero $z_2{}^a$ of $F(z) - a$, and so on. The radius of the circle Γ^p obtained after p operations is $d_p|\sigma|^{n_p}$ and its centre $z_0\sigma^{n_p}$; and so we see that the ratio

$$z_p{}^a / z_0\sigma^{n_p}$$

tends to 1 as p tends to infinity.

The sequence of points $z_0\sigma^n$ *approaches asymptotically to an infinite sequence of zeros of* $F(z) - a$, *for all values of a differing from the exceptional value* a_{z_0}.

It is clear that if there is a value a_0 exceptional P, then a_{z_0} is

equal to a_0. But in that case the result of chapter IV is more precise than the one just proved.

10. The set of points at which the family $F(z\sigma^n)$ is not normal. — Theorem 48 applies to every point z_0 of the annulus C at which the family $\{F_n(z)\}$ is not normal. The dependence of the points z_0 on the annulus C is only apparent. If $R_1\sigma$ (or R_1/σ) is substituted for R_1, z_0 is transformed into $z_0\sigma$ (or z_0/σ). The set of points $E(\sigma)$ at which the sequence $F_n(z)$, ($n > p$ if $|z| < R|\sigma|^{-p}$), is not normal, is therefore invariant for the substitution $(z, z\sigma)$ so that we might restrict ourselves, as we have done hitherto, to a consideration of that part of the set lying in a single annular region of breadth $|\sigma|$.

$E(\sigma)$ *is a closed set.* — This is its only general property, and it follows from the definition of the points at which the family is normal. These points are not isolated, for each one can be enclosed in a circle of positive radius in which the family is once more normal. The points at which the family is normal therefore constitute domains and it follows that a limiting point of non-normal points cannot be a normal point.

There may be only one point of $E(\sigma)$ *in an annulus* C. — Let us consider the integral function of zero order defined by

$$f(z) = \prod_1^\infty \left(1 - \frac{z}{\sigma'^n}\right), \qquad (|\sigma'| > 1).$$

Since $n(r)$ is less than $K \log r$, this function satisfies condition (5, 6). It follows from theorem 38 that each zero σ'^n can be surrounded by a circle of radius $\varepsilon(n)|\sigma'|^n$, where $\varepsilon(n)$ tends to zero with $1/n$, and that outside these circles the modulus of $f(z)$ tends to infinity as $|z| \to \infty$. Put $\sigma = \sigma'$ and let C be the annular region defined by $|\sigma|^{1/2} \leqslant |z| \leqslant |\sigma|^{3/2}$; the modulus of $f_n(z) = f(z\sigma^n)$ increases indefinitely with n for all points of C exterior to any small circle of centre σ. The sequence $f_n(z)$ therefore tends uniformly to infinity throughout C, save at the point σ, which is consequently the only point of C belonging to $E(\sigma)$.

If, however, there is an exceptional value a_{z_0} which is only assumed a finite number of times in a circle Γ, of centre z_0 and radius d, by the sequence $\{F_n(z)\}$, *then this point z_0 of* $E(\sigma)$ *is certainly not isolated.*

For suppose that z_0 is an isolated point of $E(\sigma)$. The sequence $\{F_n(z)\}$ will then be normal in a circle Γ_δ of centre z_0 and radius δ, ($\delta < d$), except at the point z_0 itself. Therefore from every sequence of functions $F_n(z) - a_{z_0}$ we can select a partial sequence uniformly convergent in Γ_δ, except at z_0. But if such a sequence has for limit in Γ_δ, except at z_0, a regular function or finite constant, then it converges uniformly on the circumference of a circle concentric with and interior to Γ_δ, and therefore, by Weierstrass' theorem 5, throughout this circle. Similary, if the limit of the sequence is infinity, that of the corresponding sequence of functions $1/(F_n(z) - a_{z_0})$, which, by hypothesis, are regular if $n > N$, is zero and, for the same reason, this sequence converges uniformly to zero in a circle of centre z_0. Thus the sequence of functions $F_n(z) - a_{z_0}'$ converges uniformly to infinity in a circle about z_0. So in every case uniform convergence in the neighbourhood of z_0 implies convergence at the point z_0 itself. The sequence $F_n(z)$ cannot be normal in the neighbourhood of z_0 without being normal at z_0. Thus the point z_0 of $E(\sigma)$ cannot be isolated.

In particular this is true for all the points of $E(\sigma)$ *when* $F(z)$ *has a value exceptional* P.

When the function $F(z)$ *is exceptional* P *we see that* $E(\sigma)$ *is a perfect set* (closed and dense in itself).

Whenever $F(z)$ *has a path of finite indetermination* (and a fortiori a path of finite determination, or a value exceptional P), *then there is a continuous set belonging to* $E(\sigma)$; *or more precisely, there is at least one point of* $E(\sigma)$ *on every curve surrounding the origin.*

Consider a simple curve C surrounding the origin and let D be a domain containing C (for instance D might be the domain swept through by a small circle whose centre described C). Now the maximum modulus of $F_n(z)$ in D tends to infinity, so that the family $\{F_n(z)\}$ can only be normal in D if the sequence $\{F_n(z)\}$ tends uniformly to infinity. But, since $F(z)$ is bounded on a certain receding path, every function $F_n(z)$ is less in modulus than a

number K, independent of n, in at least one point of C. Hence the functions $F_n(z)$ cannot tend to infinity. There is then at least one point of D at which the family $\{F_n(z)\}$ is not normal. But the domain D is purely arbitrary so long as it contains C, and it follows that there is at least one point P of C at which the family is not normal.

That the set E(σ) may comprise certain continuous curves is shewn at once by quite trivial examples. For instance, if we take $f(z) = e^z$ and σ real and positive, then the sequence $\{f_n(z)\}$ converges uniformly to zero in the domain to the left of the imaginary axis and to infinity in the domain to the right. The sequence is therefore not normal at points of the imaginary axis. For in a circle with its centre on this axis every sequence of functions $f_n(z)$ converges to infinity in the right-hand semi-cercle and to zero in the left-hand semi-circle. Here the set E(σ) consists of the points of the imaginary axis.

Finally it is clear that the set E(σ) is quite definitely dependent on the value of σ. A closer study of the function e^z or of the function of zero order employed above will disclose changes in the constitution of E(σ) following changes in σ.

11. Some generalisations. — Consider a function F(z) which assumes the value 0 an infinity of times (if 0 is a value exceptional P or B we replace F(z) by F(z) — a). From the sequence $\{a_n\}$ of the zeros of F(z) we can always select another sequence

$$a_{n_1}, \quad a_{n_2}, \quad \ldots\ldots, \quad a_{n_p}, \quad \ldots\ldots$$

such that, r_{n_p} being the modulus of a_{n_p},

$$(6, 11) \qquad \lim_{p=\infty} \frac{r_{n_{p+1}}}{r_{n_p}} = k > 1.$$

Now consider the sequence of functions

$$F_p(z) = F(z a_{n_p})$$

in the annular region D, $1 - \delta \leqslant |z| \leqslant 1 + \delta$. As in paragraph 9, the only possible limiting value for a sequence of functions $F_n(z)$

uniformly convergent in this annulus is infinity and, since the value of $F_n(z)$ is 0 for $z = 1$, the sequence cannot converge to infinity at all points of the annulus D. So the sequence is not normal in this annulus. Thus there is at least one point of D, and since δ is arbitrary it follows that there is at least one point $e^{i\psi}$ on the circle $|z| = 1$, at which the sequence is not normal. In a small circle described about this point as centre the functions $F_p(z)$ assume every value, with only one possible exception, an infinity of times.

If Γ_p is the circle of radius $r_{n_p}d$ and centre $a_{n_p}e^{i\psi}$, the function $F(z)$ assumes every value, save possibly a single exceptional value, an infinity of times in the domain covered by the sequence of circles Γ_p. Moreover we can choose d sufficiently small to ensure that these circles do not overlap.

Thus, corresponding to every sequence of zeros a_{n_p} of $F(z)$ satisfying condition (6, 12), we can find a sequence of circles with centres $a_{n_p}e^{i\psi}$ and radii $r_{n_p}d$, d being arbitrarily small, such that in the domain covered by these circles $F(z)$ assumes every value, with one possible exception, an infinity of times.

Now let us consider a receding curve C with an equation of the form $\varphi = \chi(r)$, ($\varphi = \arg z$), $\chi(r)$ being continuous and one-valued for all positive values of r; and let us denote by C_θ the curve obtained by rotating C about the origin through an angle θ. We shall assume that $\chi(1) = 0$, but this does not in any way restrict the generality of the set of curves C_θ. The equation of C_θ is now $\varphi = \chi(r) + \theta$. It is plain that only one of these curves C_θ passes through any given point of the plane. Let C_n be the curve passing through a_n, a zero of $F(z)$. The sequence of points in which the curves C_n cut the unit circle has at least one limiting point. Let ψ' be the argument of such a point. The curve $C_{\psi'}$, $\varphi = \chi(r) + \psi'$, is thus as close as we please to a sequence of zeros a_{n_p} of $F(z)$. We can apply the preceding proposition to these zeros. The circles Γ_p corresponding to these zeros and to a number d are, for sufficiently large values of p, interior to the domain swept through by a circle whose centre describes the curve $C_{\psi'+\psi}$ and whose radius at the point of modulus r is δr, $\delta > d$. Thus our final result, which may be stated as follows, generalises that of paragraph 8 and completes Picard's theorem :

Theorem 49. — *If C is any receding path and D the domain swept through by a circle describing C and subtending an arbitrarily small constant angle at the origin, and if* F(*z*) *is a function with an isolated essential singularity at infinity, then we can rotate* D *about the origin into a new position in which it contains an infinity of zeros of all the functions* F(*z*) — *a*, *save possibly for a single exceptional value of a.*

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§ 2. Borel **3**.
§ 3. Landau **2**.
§ 4. Bieberbach **1**; Valiron **7** and **10**.
§ 5. Montel **1**.
§§ 6, 7. Montel **2** and **3**.
§ 8. Lindelöf **6**; Montel, **2**, **3**; Julia **1**.
§§ 9, 10, 11. Julia **1**, **2** and **3**.

APPENDIX A

A note on algebraic functions.

Certain results from the theory of algebraic functions were stated without proof in chapter IV (§ 5). The object of the present note is to shew how these necessary results are obtained.

As in chapter IV consider the equation

$$\sum_{0}^{M} a_m x^{q_m} y^m = 0, \qquad (q_m > 0)$$

where

$$a_m = A_m\left(1 + \frac{K}{|x|} + \frac{K'}{|y|^{\delta}}\right) \qquad (\delta > 0).$$

Our purpose is to discuss those solutions $y(x)$ of this equation which tend to infinity as x tends to infinity.

Writing

$$Y = \frac{1}{y}, \qquad X = \frac{1}{x}$$

we are led to an equation of the form

$$\text{(A, 1)} \qquad \sum_{0}^{M} b_m X^{p_m} Y^m = 0,$$

where

$$b_m = B_m(1 + K|X| + K'|Y|^{\delta}),$$

and the problem is reduced to that of discussing those solutions $Y(X)$ of (A, 1) which tend to zero as X tends to the origin. The solution is provided by Weierstrass' classical theorem on implicit functions.

First let us consider the equation

$$(A, 2) \qquad \sum_{0}^{M} B_m X^{p_m} Y^m = 0.$$

By the hypothesis of chapter IV (§ 5), $Y = 0$ is a root of (A, 2) for $X = 0$. There are therefore a certain number P of solutions which are zero at the origin. By Weierstrass' theorem (See Goursat, *Cours d'Analyse*, third edition, vol. II, pp. 287 et seq.) these solutions are of the form

$$(A, 3) \qquad Y(X) = c_1 X^{s/q} + c_2 X^{(s+1)/q} \ldots, \qquad (q \leqslant M)$$

where q and s are integers. This theorem gives the complete formal solution of (A, 2), but in order to discuss the asymptotic behaviour of Y(X) and of the corresponding solutions of (A, 1) in the neighbourhood of the origin we require to be able to find s/q and c_1. For the purpose of finding q Puiseux [1] devised the following procedure:

Substituting the expansion (A, 3) in the left-hand side of (A, 2) we obtain an expression which is identically zero. Now

$$B_m X^{p_m} Y^m = c_1^m B_m X^{p_m + \frac{sm}{q}} + m c_1^{m-1} c_2 B_m X^{p_m + \frac{sm+1}{q}} + \ldots,$$

so that to find the terms of lowest degree we need only consider the sum

$$(A, 4) \qquad \sum_{0}^{M} c_1^m B_m X^{p_m + s\frac{m}{q}}$$

If ω is the smallest of the numbers $p_m + s\dfrac{m}{q}$ there must clearly be at least two values of m for which

$$(A, 5) \qquad p_m + s\frac{m}{q} = \omega,$$

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[1] Puiseux, *Journal de Mathématiques* (Liouville), 1850.

or the terms of lowest degree cannot annul one other. For all other values m' of m we have

$$p_{m'} + s\frac{m'}{q} > \omega = p_m + s\frac{m}{q}$$

or

$$p_{m'} - p_m > (m' - m)\left(-\frac{s}{q}\right).$$

Therefore if we plot the points of coordinates m, p_m, the points m', $p_{m'}$ lie above the line of slope $\left(-\frac{s}{q}\right)$ passing through the points for which (A, 5) is satisfied. It follows that *if a Newton's polygon* π *be constructed with the points* m, p_m *(the polygon being convex downwards and having certain of these points as vertices while the remainder lie above it) the slopes of its sides give the possible values of* $-s/q$.

We see that the values of $-s/q$ obtained by this method do not depend on the particular values of the coefficients B_m, but only on the fact that these coefficients are not zero. The numbers c_1 can be found by equating to zero the sum of the terms in (A, 4) corresponding to the selected side of π and solving this as an equation in c_1. The number of the possible pairs of numbers s/q, c_1 so obtained (each pair occuring with a frequency equal to the order of multiplicity of c_1) is equal to the number P of solutions of (A, 2) which vanish at the origin.

Now consider an equation of the form

$$\text{(A, 6)} \qquad \sum_{0}^{M} B_m(1 + t_m)X^{p_m}Y^m = 0,$$

where t_m is a function of X tending to zero with X. For $X = 0$ this equation has, as we have seen, a certain number, say P, solutions equal to zero. There are therefore exactly P solutions which tend to zero as X tends to zero [1]. To investigate these solutions let us for the moment regard the numbers t_m as equal to zero and consider a possible pair of numbers s/q, c_1 found by Puiseux's method.

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[1] See Goursat, vol. II, p. 280.

The number c_1 is determined by an equation of the form

$$\sum B_m c_1^m = 0. \tag{A, 7}$$

Now in (A, 6) write

$$Y = c_1 X^{s/q}(1 + U) \tag{A, 8}$$

and take out the factor X^ω (defined above) from the left-hand side. We then obtain an algebraic equation in U with roots that vanish for $X = 0$. In fact the multiplicity of the root $U = 0$ is equal to the multiplicity μ of c_1 as a root of (A, 7). Therefore this equation in U has again exactly μ solutions tending to zero with X. This applies to all P possible values of c_1 and s/q, so that all the P solutions of (A, 6) which vanish at the origin are effectively of the form (A, 8).

This argument remains valid if the numbers t_m are only defined in certain annular regions of centre $X = 0$, and the solutions in each of these regions are again of the form (A, 8).

Now let us consider an equation of the form

$$\sum_{0}^{M} b_m X^{p_m} Y^m = 0, \tag{A, 9}$$

where b_m is a function of X and Y which tends to B_m as X and Y tend to zero and where the equation *is known to have* a solution Y(X) tending to zero as X tends to zero (this solution may be defined only in certain annuli). On substituting Y(X) for Y in b_m we are led to an equation of the form (A, 6), whose solutions we have already discussed. For every value of X it defines a function Y(X) of the form (A, 8) : that is to say of the form

$$Y = c_1 X^{s/q}(1 + \varepsilon(X)),$$

c_1 and s/q being one of the pairs of numbers found by means of Puiseux's polygon (plainly the same pair for all values of X if it is assumed that Y is continuous. If this assumption is not made c_1 and s/q may vary with X).

Finally it follows without difficulty from this result that, when b_m is of the form indicated at the beginning of this note, we have

$$(A, 10) \qquad |U| < K\,|X|^{\delta'} \qquad (\delta' > 0)$$

whence it follows that $\varepsilon(X)$ satisfies a similar inequality. The properties assumed in chapter IV are thus established.

APPENDIX B

The genus of a function of integral order.

1. It was shewn in chapter III that the genus of a function of order ρ is not completely determined when ρ is an integer and the condition (3, 10) p. 60 is satisfied. At present our knowledge on this subject is incomplete, but we have thought it worth while to discuss a criterion for the convergence of the series

$$\sum_{1}^{\infty} r_n^{-\rho} \tag{B, 1}$$

and to prove a new result concerning the coefficients in the Taylor series. Our argument depends on the following proposition which is a modification of Jensen's inequality (3, 2) :

LEMMA I. — *For an integral function $f(z)$ we have the inequality*

$$\int_{\alpha}^{r} \frac{n(x)}{x^{1+k}}\, dx < K + \frac{\log M(r)}{r^k} + k \int_{\alpha}^{r} \frac{\log M(x)}{x^{1+k}}\, dx \tag{B, 2}$$

where k and α are positive and K *is a constant.*

Write

$$V(x) = \frac{1}{2\pi} \int_{0}^{2\pi} \log |f(xe^{i\theta})|\, d\theta . \tag{B, 3}$$

Then Jensen's equation (3, 1) can be written in the form

$$\int_{\alpha}^{r} n(x)\,dx/x = V(r) - V(\alpha),$$

where $n(x)$ is the number of zeros of $f(z)$ in the circle $|z| \leqslant x$. Now an integration by parts gives

$$\int_\alpha^r \frac{dy}{y^{1+k}} \int_\alpha^y \frac{n(x)}{x}\, dx = -\frac{V(r)-V(\alpha)}{kr^k} + \frac{1}{k}\int_\alpha^r \frac{n(x)}{x^{1+k}}\, dx,$$

so that

$$\int_\alpha^r \frac{V(x)}{x^{1+k}}\, dx = \frac{V(\alpha)}{k\alpha^k} - \frac{V(r)}{kr^k} + \frac{1}{k}\int_\alpha^r \frac{n(x)}{x^{1+k}}\, dx.$$

But we know that $V(x)$ is less than $\log M(x)$. This establishes the inequality (B, 2).

On p. 52, we obtained the following proposition

LEMMA II. — *The necessary and sufficient condition that the series* (B, 1) *should be convergent is that the integral*

(B, 4) $$\int_\alpha^X \frac{n(x)}{x^{1+\rho}}\, dx$$

should be bounded.

Now if

(B, 5) $$\int_\alpha^r \frac{\log M(x)}{x^{1+\rho}}\, dx$$

is bounded, the integral

$$\int_r^{2r} \frac{\log M(x)}{x^{1+\rho}}\, dx > \frac{\log M(r)}{\rho r^\rho}\left(1 - \frac{1}{2^\rho}\right)$$

tends to zero as r tends to infinity. Therefore $\log M(r)/r^\rho$ tends to zero. Putting $\rho = k$ in (B, 2) it follows that *if* (B, 5) *is bounded, then the series* (B, 1) *is convergent.*

The converse of this is not true when ρ is an integer, though it is not difficult to prove (using Borel's inequality (3,6)) when this is not so. We can however deduce a complementary proposition for functions of integral order from theorem (25). With our previous notation we have

$$W(r, a, b) = \int_\alpha^r \frac{n(x, a) + n(x, b)}{x}\, dx > H \log M(kr), \qquad (\alpha > 0)$$

where k is a positive number less than 1. So

$$\int_\beta^R W(r, a, b)\frac{dr}{r^{1+\rho}} > Hk^\rho \int_{\beta k}^{Rk} \frac{\log M(r)}{r^{1+\rho}}\, dr, \qquad (\beta > 0)$$

after making a change of variable on the right. Integrating by parts on the left we deduce the inequality

(B, 6)
$$\int_\beta^R \frac{n(x, a) + n(x, b)}{x^{1+\rho}}\, dx > \frac{W(R, a, b)}{R^\rho} + \rho H k^\rho \int_{k\beta}^{kR} \frac{\log M(x)}{x^{1+\rho}}\, dx - K(\alpha, a, b),$$

where K is a constant. If the integral (B, 5) is divergent, then so is the integral on the left-hand side of (B, 6) and it follows that

$$\int_\beta^R \frac{n(x, y)}{x^{1+\rho}}\, dx$$

cannot be bounded for two distinct values of y. If by $r_n(x)$ we denote the modulus of the $n'th$ zero of $f(z) - x$, the series

(B, 7)
$$\sum_1^\infty \frac{1}{r_n(x)^\rho}$$

cannot converge for more than one value of x.

THEOREM 1. — *If $f(z)$ is an integral function of order ρ and the integral* (B, 5) *is bounded, then the series* (B, 7) *is convergent for all values of x. All the functions $f(z) - x$ are of genus $\rho - 1$* [1].

If the integral (B, 5) *is unbounded the series* (B, 7) *is divergent save possibly for a single exceptional value of x. Except for this value of x all the functions $f(z) - x$ are of genus ρ.*

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[1] By the proposition preceding theorem 28, on p. 90.

2. In this theorem the condition of convergence or divergence of

$$\int_\alpha^\infty \frac{\log M(x)}{x^{1+\rho}}\, dx$$

may be replaced by the condition of convergence or divergence of the series

$$\text{(B, 8)} \qquad \sum_1^\infty \frac{1}{R_n^{\rho}},$$

where R_n is the rectified ratio defined in chapter II. The transformation used in Lemma II shews that the necessary and sufficient condition for the convergence of (B, 8) is that

$$\int_\alpha^r \frac{N(x)}{x^{1+\rho}}\, dx$$

should be bounded, $N(x)$ being the number defined in chapter II. For, integrating by parts, we have

$$\int_\alpha^r \frac{dx}{x^{1+\rho}} \int_\alpha^x \frac{N(y)}{y}\, dy = K - \frac{1}{\rho r^\rho}\int_\alpha^r \frac{N(y)}{y}\, dy + \frac{1}{\rho}\int_\alpha^r \frac{N(x)}{x^{1+\rho}}\, dx$$

and, since

$$\log M(r) \backsim \int_\alpha^r N(x)\, dx/x,$$

this gives

$$\int_\alpha^r \frac{\log M(x)}{x^{1+\rho}}\, dx + \frac{\log M(r)}{\rho r^\rho} = h(r)\left[K + \frac{1}{\rho}\int_\alpha^r \frac{N(x)}{x^{1+\rho}}\, dx\right]$$

where $h(r)$ is a number confined between two fixed positive limits. A repetition of a former argument then proves our assertion. Thus :

THEOREM 2. — *If* ρ *is an integer, the convergence of* (B, 8) *implies the convergence of* (B, 1) *and its divergence implies the divergence of* (B, 7) *for all values of* x *save possibly a single exceptional value.*

3. We require the following lemma :

Lemma III. — *If all the numbers a_n are positive and if the series*

$$\sum_1^{\infty} a_n$$

is convergent, then the series

$$\sum_1^{\infty} \sqrt[n]{a_1 a_2 \ldots a_n}$$

is also convergent.

Writing $b_n = (a_1 a_2 \ldots a_n)^{\frac{1}{n}}$ we have, if $k > 0$,

$$\sum_1^n b_m = \sum_1^n \left[\prod_1^m (s^k a_s)\right]^{\frac{1}{m}} (m\,!)^{-\frac{k}{m}} \leqslant \sum_1^n \left[Km^{-k} \frac{\sum_1^m s^k a_s}{m}\right].$$

In this last series the coefficient of a_s is less than

$$Ks^k \left(\frac{1}{s^{k+1}} + \frac{1}{s^{k+2}} + \ldots\right) < K.$$

Therefore [1]

$$\sum_1^n b_m < K \sum_1^n a_m.$$

It follows, since the numbers R_n do not decrease, that the series

$$\sum_1^{\infty} R_n^{-\rho} \qquad \text{and} \qquad \sum_1^{\infty} (R_1 \ldots R_n)^{-\rho/n}$$

converge and diverge together.

Now

$$e^{G_n} = R_1 R_2 \ldots R_n.$$

(1) For this simple argument we are indebted to Mr. Littlewood.

and consequently we may replace the conditions of convergence and divergence of the series (B, 8) by the same conditions for the series

$$\text{(B, 9)} \qquad \sum_{1}^{\infty} e^{-G_n \rho/n}.$$

In particular we have the following result :

THEOREM 3. — *If $f(z)$ is of integral order ρ and such that the series*

$$\sum_{1}^{\infty} |c_n|^{\rho/n}.$$

is divergent, then the function $f(z) - x$ is of genus ρ for all values of x, save possibly a single exceptional value.

REFERENCES

§§ 1 and 2. Valiron **12** and **15**.
§ 3. Original results due to E. F. Collingwood.

APPENDIX C

On the zeros of functions of integral order and regular growth.

By R. C. YOUNG.

In pursuing our investigations with regard to the distribution of the values of a function $f(z)$ of finite order ρ, we indicated in Chapter III a method of procedure by which it is possible to deduce a number of results (some of which we stated) concerning the asymptotic([1]) relation between the number of zeros of $f(z) + x$ in the circle $|z| \leqslant r$, — which number we denote by $n(r, x)$ as before, — and the maximum modulus $M(r)$ of $f(z)$ on the circumference.

We saw that by supposing the order ρ not to be an integer, we could obtain closer approximations (notably in the case of regular growth), the methods failing when ρ was an integer. Though this difference between the two cases of finite order belongs partly to the nature of things and is not merely a consequence of imperfect methods, it would be interesting to know in what measure results such as follow, for instance, from the method of p. 70 when ρ is not an integer, may be extended to comprise the case when ρ is an integer, — generalising if need be the notion of exceptionality for a given value x.

We are about to show how, by a very simple transformation, given a function $f(z)$ of positive integral order ρ, we may form another integral function, with zeros corresponding exactly to those of $f(z)+x$, but with non-integral order, and with its maximum modulus in extremely close relation with that of $f(z)$ for all values of x except possibly some belonging to a set whose projection on any straight line is of zero linear content.

The results we shall deduce therefrom are the following, true *when ρ is any positive integer*.

([1]) With respect to r. It is understood throughout that x is given fixed values.

THEOREM C 1. — *If the ratio* $\log_2 M(r)/\log r$ *has a limit* (regular growth), *then the ratio* $\log n(r, x)/\log r$ *has the same limit, except possibly for some values of* x *belonging to a set whose projection on any straight line is of zero content.*

THEOREM C 2. — *If the ratio* $\log M(r)/r^{\rho}$ *is bounded positively* (1) (very regular growth) *then so is the ratio* $n(r, x)/r^{\rho}$, *except possibly for values of* x *belonging to a set whose projection on any straight line is of zero content*

The interest of these propositions lies in the fact that in some examples of functions of regular and very regular growth, there are actually more than a countable set of values of x for which the ratios $\log n(r, x)/\log r$ and $n(r, x)/r^{\rho}$ do not have the properties in question.

1. The excepted set of points. — Suppose $a_1(x), a_2(x), \ldots, a_n(x), \ldots$ denote polynomials of degrees not exceeding q and let the typical zero of $a_n(x)$ be denoted by $\alpha_{n,i}$. About each zero $\alpha_{n,i}$ as centre we draw a circle of radius $d_n = K^{-n}$ and denote by Γ_K the set of the internal points of these circles : the projection on any straight line of this set Γ_K has a linear content not exceeding $2q/(K-1)$.

If we now make K tend to infinity, the variable set Γ_K reduces to its inner limiting set or i-set, constituted by all the points internal to an infinite number of Γ_K, and comprising therefore the countable set of points $\alpha_{n,i}$ together with some (or all) of its limiting points. It is of zero plane content and its projection on any straight line is of zero linear content. It is not necessary to specify its properties more nearly, though we may remark that not only the sum of the diameters of the circles which constitute Γ_K tends to o with K^{-1}, but also the sum of the i^{th} roots of these diameters, where i may be any number.

By leaving out whatever isolated points the i-set may contain (and which can only be some or all of the points $\alpha_{n,i}$), we get the type of set which we shall have at most to except in our result.

(1) More precisely, if it has its limits of indetermination for infinite r comprised between positive limits.

We will speak of the set thus defined by means of the polynomials $a_n(x)$ as associated to the polynomials $a_n(x)$.

2. Fundamental transformation. — Consider the function

$$(f(z)+x)\,(f(z\tau)+x)\;\ldots\ldots\;(f(z\tau^{q-1})+x)$$

where q is an integer chosen greater than ρ for our purposes, and where τ is a primitive q^{th} root of 1.

Transformed to the plane $Z=z^q$, this function, denoted henceforth by $F(Z, x)$, is still an integral function by the properties of the q^{th} roots of 1, and has exactly the corresponding zeros in the Z-plane to those of $f(z)+x$ in the z-plane, with the same multiplicity, so that we have $n(r, x) = N(R, x)$, $(R = r^q)$.

PROPOSITION. — *For every x, except possibly for values belonging to a set whose projection on any straight line is of zero content, there exists between the maximum moduli of $F(Z, x)$ and $f(z)$, — denoted respectively by $M(R, x)$ and $M(r)$, — a relation of the form*

$$(C, 1) \qquad \log M(R, x) = h_1 \log M(h_2 r) \qquad (R = r^q)$$

h_1 and h_2 representing two positive functions of r, depending on x, with finite positive limits of indetermination for infinite r.

Before proceeding to prove this proposition, we may at once remark that, for any value of x for which it holds, the ratios $\log_2 M(R, x)/\log R$ and $\log_2 M(r)/q \log r$ have the same limits of indetermination, so that in particular $F(Z, x)$ is then effectively of order ρ/q.

Moreover for such values of x, all properties of regular growth will be simultaneous to $f(z)$ and $F(Z, x)$; hence by the propositions stated for non-integral order on pp. 70-71, the theorems C 1 and C 2 are immediate consequences of (C, 1), with the conditions under which it is stated to hold.

All turns therefore on the proof of this relation.

3. The relation (C, 1) virtually represents two inequalities true for large values of r, — more precisely, we should have, for $r > r_r$:

$$\text{(C, 1a)} \qquad k_1 \log M(k_2 r) < \log M(R, x) < k_1' \log M(r)$$

the k's being positive functions of x only, finite everywhere in the region specified.

Now, as immediately follows from the definition of $F(Z, x)$:

$$\text{(C, 2)} \qquad \log M(R, x) < \Delta q \log M(r)$$

where $\lim\limits_{r=\infty} \Delta = 1$ for every x.

There remains therefore only to establish the first of the two inequalities (C, 1 a).

To do this, we observe first that if $G(R) = \Sigma C_n R^n$ be defined as having for its n^{th} coefficient C_n the largest of the moduli of the coefficients of x in $a_n(x)$ (the latter polynomial being the coefficient of Z^n in $F(Z, x)$), then the ratio $\log G(R)/\log M(r)$ is bounded positively below.

This is easily seen by introducing the coefficients $A_i(Z)$ of $F(Z, x)$ when put in the form of a polynomial in x. $A_i(Z)$, if taken to denote the coefficient of x^{q-i} in $F(Z, x)$, is the sum of

$$\frac{q(q-1)\ldots(q-i+1)}{1 \cdot 2 \ldots i}$$

products of i of the functions $f(z\tau^k)$, and so the absolute value of A_i is less than for instance $[qM(r)]^i$.

On the other hand, as the value $x = -f(z)$, if not zero, satisfies the equation $\dfrac{F(Z, x)}{x^{q-1}} = 0$, giving in particular

$$|x| < \sum_{i=1}^{i=q} \left| \frac{A_i}{x^{i-1}} \right|$$

we have by the above, putting $|x| = M(r)$,

$$M(r) < \sum_{i=1}^{i=q} q^{i-1} |A_i|^{1/i}$$

for some value of Z with $|Z| = R = r^q$.

Now with our definition, $G(R)$ is a dominant of all the $A_i(Z)$: in fact C_n is the largest of the moduli of the respective n^{th} coefficients of these functions. Also $G(Z)$, like $A_i(Z)$, is an integral function ([1]). We may therefore, assuming $R > R_0$ to insure $G(R) > 1$, write

$$M(r) < \sum_{i=1}^{i=q} q^{i-1} G(R).$$

Hence

$$\log G(R) > \Delta \log M(r) \tag{C, 3}$$

where $\lim_{r=\infty} \Delta = 1$.

To establish the first inequality (C, 1 a), we may henceforth substitute in it $G(R)$ for $M(r)$, reducing our problem to the proof of an inequality of the form

$$\log M(R, x) > k_1 \log G(k_2 R). \tag{C, 4}$$

By the properties of the maximum term of a function of finite order (such as $G(Z)$ was seen to be), such an inequality certainly holds whenever between the respective coefficients of $G(Z)$ and $F(Z, x)$ one of the form

$$|a_n(x)| > k_1 C_n (k_2)^n \tag{C, 5}$$

holds for all $n > n_x$, — k_1 and k_2 not depending on n, but only on x.

For by multiplying by $|Z|^n = R^n$ and choosing n so that $C_n(k_2 R)^n$ is the maximum term of $G(k_2 Z)$ ([2]), we see that (C, 5) involves

$$\log M(R, x) > \log |a_n(x) Z^n| > \Delta \log G(Rk_2) \qquad (R > R_x)$$

with $\lim_{R=\infty} \Delta = 1$.

To obtain an inequality of the form (C, 5), we use the two facts that

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(1) See e. g. theorem 14, p. 40.

(2) We have to take R sufficiently great to insure this maximum term having a rank $> n_x$.

a) if $|x'| > \frac{x}{2}$ and $|a_n(x')|$ is the maximum modulus of $a_n(x)$ for the radius $|x'|$, then by Cauchy's theorem

$$|a_n(x')| \geqslant C_n |x'|^{\lambda_n} > C_n \left|\frac{x}{2}\right|^{\lambda_n}$$

where λ_n is some positive integer or zero and does not exceed $q-1$.

b) if $|x'| < x$ and both x', x are at distances not less than d_n from the zeros of a_n and from the origin, then (¹)

$$\left|\frac{a_n(x)}{a_n(x')}\right| > \left|\frac{d_n}{3x}\right|^{q-1}.$$

Now provided $4qd < |x|$, we can certainly for each n find a circle of radius between $\left|\frac{x}{2}\right|$ and $|x|$ (the centre being at the origin), which is completely external to the small circles of radius d_n round the zeros of $a_n(x)$. So we are assured that we can for each n find a point satisfying the conditions required for x' and we may therefore write for each $n \geqslant 1$

$$|a_n(x)| > C_n d_n^{q-1} 6^{1-q} x^{-\lambda}$$

where λ is $q-1$ or 0 according as $x \geqslant 1$ or $x < 1$.

If Γ_K denotes the set of points ξ defined as in n° 1 by $|\xi - \alpha_{n,i}| < K^{-n}$ $(i = 1, 2, \ldots, \mu_n;\ n = 1, 2, \ldots)$ and E be its i-set for infinite K, then given any value x not belonging to E we can determine K_x so that $|x - \alpha_{n,i}| > K_x^{-n}$ $(i = 1, 2, \ldots, \mu_n;\ n = 1, 2, \ldots)$ and $4qK_x < |x|$.

(¹) Inequalities of this form are easily proved by splitting a_n into its factors $(x - \alpha_{n,i})$ and considering separately e. g. those for which $|\alpha_{n,i}| \leqslant 2x$, and $|\alpha_{n,i}| > 2x$. In the latter case we write

$$|x' - \alpha_{n,i}| < |\alpha_{n,i}| + |x| = [|\alpha_{n,i}| - |x|]\left\{1 + \frac{2|x|}{|\alpha_{n,i}| - |x|}\right\} < 3[|\alpha_{n,i}| - |x|]$$

and a fortiori

$$\left|\frac{x' - \alpha_{n,i}}{x - \alpha_{n,i}}\right| < 3 < \left|\frac{3x}{d_n}\right|.$$

For all such values of x we may therefore, substituting K_x^{-n} for d_n in the above, write for every $n \geqslant 1$

$$(C, 5a) \qquad |a_n(x)| > h(x) C_n (K_x^{1-q})^n,$$

$h(x)$ not depending on n.

If x be an isolated point of the i-set, denote by a_{n_x} the polynomial of highest rank of which it is a zero; by leaving out of account the n_x first polynomials, we can still define K_x so that an inequality of the form (C, 5a) holds for all $n > n_x$.

We saw that this sufficed to deduce an inequality of the form (C, 4), which with (C, 3) and (C, 2) proves our relation (C, 1), the possible exceptions belonging to a subset[1] of the set defined by means of the coefficients $a_n(x)$ in the manner explained in § 1.

4. Remark on the formation of examples with exceptional values. — Without going into details, the following brief indication may be of help to the reader.

Given a set of finite numbers θ_n, we define the auxiliary function of z

$$\Psi_x(z) = x + \sum_{1}^{\infty} \frac{x - \theta_n}{n!} z^n .$$

If x_0 is not a limiting point[2] of the set $\theta_n (n = 1, 2, \ldots)$, the function $\Psi_{x_0}(z)$ has its maximum modulus asymptotically equal to e^r [3]. Referring to theorem C 2, we choose θ_0 so that $\Psi_{x_0}(z) - \theta_0$ has the number of its zeros within $|z| \leqslant r$ asymptotically greater than Kr. We then define

$$f(z) = -e^{-z} \sum_{0}^{\infty} \frac{\theta_n}{n!} z^n .$$

The function $f(z) + x_0$ has the same zeros as $\Psi_{x_0}(z) - \theta_0$, and so $f(z)$

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(1) On closer examination, we should find that there are other points of the i-set, besides its isolated points, for which a relation of the form (C, 1) may still be proved to hold. It is not necessary, for instance, to consider all the polynomials a_n for $n > n_x$.

(2) In the extended sense, including values assumed by θ_n for an infinite sequence of indices n.

(3) By the proposition of p. 44, with Theorem 14.

has the logarithm of its maximum modulus $M(r)$ asymptotically greater than Kr (1). As $\log M(r)$ is evidently less than $2r$, *the function $f(z)$ is of very regular growth and order* 1. For values of x which we may call ordinary for $f(z)$, with reference to theorem C2, the ratio $n(r, x)/r$ is bounded.

It is possible so to choose the numbers $\theta_n (n \geqslant 1)$ that when ξ belongs to certain of their limiting points, the auxiliary function $\Psi_\xi(z)$ has its maximum modulus less than e^{r^α} for a sequence of values of r, where α is less than 1, involving by formula (3, 2), for $f(z)+\xi = e^{-z}(\Psi_\xi(z) - \theta_0)$,

$$\underline{\lim} \frac{\log n(r, \xi)}{\log r} \leqslant \alpha,$$

ξ is then an exceptional value for $f(z)$.

For instance if in a sequence of intervals

$$N_q \leqslant n < N_q' \qquad (N_{q+1} > N_q' > N_q)$$

the numbers θ_n are all equal to ξ_1, and if $(N_q' - N_q)$ is chosen sufficiently large, we may so arrange that the maximum modulus of $\Psi_\xi(z)$ is less than $e^{\sqrt{r}}$ for instance, for a sequence of values of r.

If we choose a sufficiently rapidly increasing sequence for $N_{q+1} - N_q'$, we can dispose of the remaining θ's so that the same circumstance presents itself for $\xi_2, \xi_3, \ldots, \xi_i$. We have then an example in which there are at least these i exceptional values.

A close examination into the sufficient conditions for ξ being an exceptional value for $f(z)$, leads to definitions of θ_n giving more than a countable number of actually exceptional values ξ, $\Psi_\xi(z)$ having its maximum modulus less than $e^{r^\alpha} (\alpha < 1)$ for a sequence of values of r.

REFERENCES

Borel **7**; Sire **3**; Valiron **9**.

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(1) For by formula (3, 2), p. 51, deduced from Jensen's theorem

$$\int_0^r \frac{n(x, x_0)}{x} dx < \Delta \log M(r), \qquad \lim_{r=\infty} \Delta = 1.$$

APPENDIX D

The inverse function of an integral function $f(z)$.

1. The problem of the paths of determination and asymptotic values of an integral function considered in chapter V can be approached from another point of view.

Let $w = f(z)$ be an integral function and $z = \Phi(w)$ be the inverse of this function. That is to say that for all values of w the function $\Phi(w)$ satisfies the equation

$$w = f(z).$$

Suppose that, in the z-plane, there is a receding path Γ of finite determination ω. It is clear that there is in the w-plane a corresponding path C proceeding to the point ω such that, as w tends to ω by this path, $z = \Phi(w)$ recedes along Γ. Now, $f(z)$ being an integral function, a finite value of w corresponds in general to a finite value of z. The point ω is therefore a singularity of the inverse function $\Phi(w)$, and it appears that we shall be able to identify the asymptotic values of $f(z)$ with certain of the singularities of $\Phi(w)$. The problem was first considered from this point of view by Hurwitz (1), and he stated that the point ω is always a transcendant singularity of the function $\Phi(w)$, and conversely that such a singularity is always an asymptotic value of $f(z)$. A rigorous proof of this theorem was given in 1914 by Iversen (2) to whom the first satisfactory systematic treatment of the subject is due. In this note we develop the theory of $\Phi(w)$ only so far as the proof the theorem we have just mentioned. Many very interesting results cognate with those of chapter V have been obtained, notably by Iversen, but we are debarred from dealing with them.

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(1) Hurwitz **3**.
(2) Iversen **1**.

2. Let $w = f(z)$ be an integral function of z and consider the planes of z and w. Let z_0 be a point at which $f'(z)$, the derivative of $f(z)$, is not zero. Then, in virtue of the relation

$$w = f(z), \tag{D, 1}$$

there corresponds to z_0 a finite value w_0, and by a well-known theorem in the theory of implicit functions, we can find two circles γ_0 and c_0 described about the points z_0 and w_0 as centres, in the planes of z and w respectively, such that, for every value of w in c_0, the equation (D, 1) has a unique root in γ_0. Further, this root is represented inside c_0 by a convergent power series ([1])

$$z = z_0 + \mathfrak{P}(w - w_0). \tag{D, 2}$$

We shall denote this series by the symbol e_{z_0} and its circle of convergence by c_{z_0}. If w is any point in c_0, then equation (D, 1) is satisfied identically on substituting for z its value (D, 2). Now (D, 1) may be written

$$f(z) - w = 0 \tag{D, 3}$$

and, since $f(z)$ is an integral function, the left-hand side of this equation is a function regular in c_{z_0}, if we substitute for z its value given by (D, 2). We know that this function vanishes identically in c_0, and it therefore follows, by a well-known theorem, that equation (D, 3) is satisfied throughout c_{z_0}. Hence in this circle c_{z_0} the series (D, 2) satisfies equation (D, 1).

Since $f(z)$ is a uniform function it follows that at distinct points in the circle c_{z_0} the series (D, 2) assumes different values. Thus the relation (D, 2) establishes a (1 : 1) correspondence between the interior of the circle c_{z_0} and a certain domain χ_{z_0} in the z-plane. No part of this domain covers any other part of it. Two corresponding points w and z always satisfy the relation (D, 1).

It is easy to see that the correspondence between c_{z_0} and χ_{z_0} is a conformal one at all interior points. For, substituting the series

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([1]) We use the symbol $\mathfrak{P}(z)$ to denote a power-series convergent in a certain circle and vanishing at $z = 0$, but not necessarily the same in every formula.

(D, 2) in equation (D, 1) and differentiating with respect to w, we find

$$f'(z)\frac{dz}{dw} = 1,$$

which shews, since dz/dw is finite, that $f'(z)$ does not vanish in χ_{z_0}.

Therefore for every value of w interior to c_{z_0} equation (D, 1) has one and only one root interior to χ_{z_0}, and this root is represented by the series (D, 2). This series we shall call the *inverse element* e_{z_0} corresponding to the point z_0. We can form such an element for every point of the set in which $f'(z)$ does not vanish. There is an infinity of such elements (we denote by e_z the element corresponding to a point z for which $f'(z) \neq 0$) and we shall shew that *together they constitute a single analytic function* $z = \Phi(w)$. This function $\Phi(w)$ *is by definition the inverse function of the integral function* $f(z)$.

Let us denote by $\mathcal{E}$ the set of elements e_z. We have then to prove two things :

(α) Every analytic continuation of an element of $\mathcal{E}$ leads to another element of $\mathcal{E}$.

(β) Every element of $\mathcal{E}$ can be obtained by continuation from a given element of $\mathcal{E}$.

We propose to adopt the following convention regarding the continuation of an element e_z : two elements e_{z_1} and e_{z_2} will be said to be *immediate continuations* of one another when their circles of convergence have a common part in which the two series assume the same values. This definition can be extended to fractional power series if one determination of one series is equal to one determination of the other in the common part of their circles of convergence.

We can prove (α) for an immediate continuation of (D, 2) to a point w_1 in the circle c_{z_0}. With w_1 the element e_{z_0} correlates a point z_1 in χ_{z_0}. Now the immediate continuation of this element at this point w_1 gives, for all values of w in a certain circle of centre w_1, the unique root of equation (D, 1) in a certain circle about z_1. But since z_1 is interior to χ_{z_0} the derivative $f'(z)$ is finite and different from zero, so that to this point z_1 there corresponds a certain element e_{z_1} of the set $\mathcal{E}$, and this element represents the same root in

a certain neighbourhood of w_1. Therefore this element is identical with the continuation of e_{z_0}.

The truth of proposition (α) follows at once, for every continuation of e_{z_0} is effected by means of a finite number of immediate continuations.

The following corollary will be useful to us in the sequel :

If two elements e_{z_1} and e_{z_2} of the set $\mathfrak{E}$ assume the same value $z = z'$ at a point $w = w'$ common to their circles of convergence, then they are immediate continuations of one another.

For z' is common to the two domains χ_{z_1} and χ_{z_2}, so that $e_{z'}$ is an immediate continuation of both e_{z_1} and e_{z_2} and it follows at once that these two elements are immediate continuations of one another.

Now consider proposition (β). Let $e_{z'}$ be any assigned element of the set $\mathfrak{E}$ distinct from e_{z_0} and join z_0 to z' by a simple curve Γ on which $f'(z) \neq 0$. Let z_1 be the first point of Γ in which this curve intersects the contour of χ_{z_0}. Corresponding to z_1 there is a certain element e_{z_1} and a domain χ_{z_1}, of which z_1 is an interior point, extending into χ_{z_0}. Let z_2 be the first point of Γ in which it intersects the contour of χ_{z_1} and let e_{z_2} and χ_{z_2} be the corresponding element and domain, the latter extending into χ_{z_1} and containing z_1, and so on.

Since any two consecutive domains of the sequence $\chi_{z_0}, \chi_{z_1}, \ldots$ have a common part it follows, from the corollary just proved, that successive elements $e_{z_0}, e_{z_1}, \ldots$ are immediate continuations of one another. It is therefore sufficient to shew that there is a finite number n such that z' lies inside the domain χ_{z_n}.

Suppose that this is not so. Then the sequence of points z_ν has a finite limit point $\bar{z}$ on Γ. Now by hypothesis $f'(\bar{z}) \neq 0$, so that this point $\bar{z}$ is interior to a certain domain $\chi_{\bar{z}}$. To the sequence of points z_ν contained in $\chi_{\bar{z}}$ there corresponds a sequence of points w_ν contained in the circle $c_{\bar{z}}$ and having the point $\bar{w} = f(\bar{z})$ as limit. Now, by the corollary, the immediate continuation of $e_{\bar{z}}$ to any one of these points w_ν is identical with the corresponding element e_{z_ν}, and it is clear that we can choose ν so large that $\bar{w}$ will lie inside c_{z_ν}. But, if this were so, the point $\bar{z}$ would lie inside the domain χ_{z_ν},

which is contrary to our hypothesis. The theorem is therefore proved.

3. The singularities of $\Phi(w)$. — We have now given a precise definition of $\Phi(w)$ which has been shewn to be consistent. Our next topic must be the singularities of the function $\Phi(w)$ thus defined.

Consider a continuous path G starting from the point w_0, and suppose that we have effected the continuation of e_{z_0} along this path. If there are no singular points of $\Phi(w)$ on G there is always a positive number ε which does not exceed the radius of convergence of any element of $\mathcal{E}$ encountered in continuation along a finite segment of G. In fact e_{z_0} can be continued indefinitely along G. If, however, there is a point ω on G such that the radius of convergence of the element e_z tends to zero as w approaches ω, then ω is a singularity of $\Phi(w)$ and the question arises as to how this function behaves as w tends to ω. Now successive elements of $\Phi(w)$ give rise to a curve Γ in the z-plane corresponding to G, and we shall shew that *as w approaches ω, z tends to some determinate value.*

The proof of this is simple. Since the zeros of an integral function are isolated points, we can find a circle C, of arbitrarily large radius, on which there is no root of the equation

$$f(z) = \omega. \tag{D, 4}$$

Let ε_1 be the minimum modulus of $f(z) - \omega$ on C. Now C contains only a finite number of roots $z_1, z_2, \ldots, z_n$, of equation (D, 4) and these points can be surrounded by small non-overlapping circles $c_1, c_2, \ldots, c_n$, all interior to C. Let ε_2 be the minimum modulus of $f(z) - \omega$ on the circumferences of these circles, and let ε be the smallest of the two numbers ε_1, ε_2. Then, if T is the domain bounded by C and the small circles $c_1, c_2, \ldots, c_n$, the function $\dfrac{1}{f(z) - \omega}$ is regular in T and so attains its maximum on the boundary. Therefore at all interior points of T we have

$$\left|\frac{1}{f(z) - \omega}\right| < \frac{1}{\varepsilon}$$

and consequently

$$|f(z) - \omega| > \varepsilon.$$

Now, by hypothesis, the element e_{z_0} can be continued along G to within an arbitrarily small distance of the point ω. It follows from the last inequality that, when w is interior to the circle $|w-\omega| = \varepsilon$, the corresponding point $z = \Phi(w)$ lies outside the domain T; that is to say, either inside a definite circle c_ν, or outside the large circle C. But this is true however large the circle C and however small the circles c_ν may be. Hence we have the following theorem :

If an element of the inverse function $z = \Phi(w)$ is continued along a path ending in a singularity ω, then z tends either to a finite root of the equation $f(z) = \omega$ or to infinity as w approaches ω.

4. Algebraic singularities. — Let us first consider the case in which z tends to a finite value ζ as w tends to ω. In the first place it is clear that $f'(\zeta) = 0$. For if not there would be a regular element e_ζ, with a circle of convergence c_ζ, corresponding to ζ. We could find a point w of G in c_ζ sufficiently near to ω to insure that the corresponding point z of Γ defined by $\Phi(w)$ would lie in the domain γ_ζ. Then, by the corollary, e_ζ end e_z would be immediate continuations of one another, so that ω would not be a singular point for the particular branch of $\Phi(w)$, which is contrary to our hypothesis.

Thus in the neighbourhood of ζ the function $f(z)$ is represented by a convergent series of the form

$$w = \omega + a_k(z-\zeta)^k[1 + \mathfrak{P}(z-\zeta)],$$

where k is an integer greater than 1. Hence

$$\text{(D, 5)} \qquad \begin{aligned}(w-\omega)^{1/k} &= a_k^{1/k}(z-\zeta)[1+\mathfrak{P}(z-\zeta)]^{1/k}\\ &= a_k^{1/k}(z-\zeta)[1+\mathfrak{P}(z-\zeta)].\end{aligned}$$

It follows from the theory of implicit functions that there is a circle c' of centre ω in the w-plane in which a power series in $(w-\omega)^{1/k}$ represents the only solutions of (D, 5) in a corresponding circle γ' of centre ζ. There are k solutions corresponding to the k different determinations of $(w-\omega)^{1/k}$. The series is of the form

$$\text{(D, 6)} \qquad z = \zeta + a_k^{-1/k}(w-\omega)^{1/k}\left[1 + \mathfrak{P}\left\{(w-\omega)^{1/k}\right\}\right],$$

whence it follows that the k branches corresponding to the k different determinations of $(w-\omega)^{1/k}$ are all different in a small circle of centre ω. The argument of § 1 shews that the series (D, 6) satisfies equation (D, 1) throughout its circle of convergence c. The function z thus defined may be regarded as uniform on a Riemann surface F of k sheets covering the circle c and having ω as a branch point of order k. The properties of F are precisely analogous to those of c_z, the circle of convergence of a regular element. To distinct points of F correspond distinct values of z, and the relation (D, 6) establishes a (1 : 1) correspondence between the interior of F and a certain domain γ of the z-plane, no two parts of which coincide. The correspondence is conformal except at the points ζ and ω. To every point $w_1 (\neq \omega)$ of F there corresponds a point z_1 at which $f'(z) \neq 0$, as can be proved by the argument of § 2. Therefore to z_1 corresponds an element e_{z_1} of $\mathcal{E}$, and it is clear, by the corollary, that this element is identical with the immediate continuation of (D, 6) to w_1. Thus every continuation of (D, 6) belongs to $\mathcal{E}$. In fact if w_1 is a point of G interior to c and so near to ω that z_1, the corresponding point of Γ, lies inside γ, then the element e_{z_1} obtained by continuing (D, 2) along G can be identified with the immediate continuation of (D, 6) to w_1. Now (D, 6) shews that ω is an *algebraic singularity* of $z = \Phi(w)$, so that we have proved that *if z tends to a finite value as w tends to a singular point ω, then this is an algebraic point of the inverse function* $\Phi(w)$.

If, on the other hand, $\Phi(w)$ tends to infinity as w tends to ω by G, then ω cannot be either a pole or an algebraic singularity of $\Phi(w)$. For if it were an algebraic singularity we should have in its neighbourhood

$$z = \frac{a}{(w-\omega)^{1/k}}\left[1 + \mathfrak{P}\left\{(w-\omega)^{1/k}\right\}\right],$$

k being a positive integer, which would imply for $f(z)$ an expansion of the form

$$f(z) = \omega + \frac{a^k}{z^k}\left[1 + \mathfrak{P}\left(\frac{1}{z}\right)\right].$$

for all sufficiently large values of z, which is plainly impossible. A similar argument shews that ω cannot be a pole.

Summing up, we see that every finite algebraic singularity of $\Phi(w)$ corresponds to a root of the equation $f'(z) = 0$, and conversely that to any root of this equation there corresponds an expansion of the form (D, 6) satisfying equation (D, 1), so that the corresponding point w is an algebraic singularity of the inverse function $\Phi(w)$.

The only finite algebraic singularities of the inverse function $\Phi(w)$ *are the points* $w^{(1)}, w^{(2)}, \ldots, w^{(\nu)}, \ldots$ *defined by the equations*

$$f'(z^{(\nu)}) = 0; \qquad w^{(\nu)} = f(z^{(\nu)}).$$

5. Transcendant singularities. — Since the point ω cannot be either a pole or an algebraic singularity when z tends to infinity as w approaches ω by the path G, it must be either an essential or a transcendant singularity of $\Phi(w)$, according as the branch of $\Phi(w)$ under consideration is or is not *indeterminate at the point* ω. To give a precise meaning to this, consider the path Γ described by z' as w tends to ω along G, and let C be a circle in the w-plane of radius r and centre ω. There is a point z_r on Γ such that the inequality $|f(z) - \omega| < r$ is satisfied at all points of Γ beyond z_r. Suppose that the element e_{z_r} is continued from the point w_r on the circumference of C by all possible paths interior to C up to the point ω. There is a set of points in the z-plane corresponding to the interior points of the circle C, and this set, together with its limit points, constitutes a connected domain Δ_r containing the curve Γ beyond the point z_r. As r tends to zero the domain Δ_r tends to a limiting domain Δ_0, which we call *the domain of indetermination* of $\Phi(w)$ at the point ω. If Δ_0 reduces to a single point we say that the branch of $\Phi(w)$ is *determinate at* ω, while it is *indeterminate* in the contrary case. With this definition we shall shew that *the branch of* $\Phi(w)$ *under consideration is always determinate at the point* ω. For Γ is, by hypothesis, a receding path on which $f(z)$ tends to ω. But we know that when $|w - \omega| < \varepsilon$, $z = \Phi(w)$ lies outside an arbitrarily large circle C. For if it did not, z would tend to a finite value as w tends to ω. Therefore, for $r < \varepsilon$, Δ_r lies entirely outside C, so that Δ_0 reduces to the point at infinity, and our assertion is proved. Hence, *if* $z = \Phi(w)$ *tends to infinity as* w *tends to* ω, *this point is a transcendant singularity of* $\Phi(w)$.

Combining this result with that of § 3 we have the following theorem :

The inverse function has no singularities other than algebraic and transcendant singularities.

We have seen that if ω is a transcendant singularity of $\Phi(w)$, then $f(z)$ has a path of determination ω. Conversely, if $f(z)$ has a path Γ of finite determination ω, this point is a transcendant singularity of $\Phi(w)$. For corresponding to Γ there is a path G proceeding to ω. If G passes through any algebraic singularity of $\Phi(w)$ it can be deformed by the introduction of small circular arcs so as to avoid these points, and then, in virtue of the properties proved in § 4, an element e_z corresponding to a point of Γ can be continued up to ω in such a way that z recedes along Γ.

We have thus proved the theorem stated in § 1 of this note :

The asymptotic values of an integral function $f(z)$ can be identified with the transcendant singularities of its inverse function $\Phi(w)$.

Many of the results proved directly in chapter V can be deduced from properties of the inverse function $\Phi(w)$, but it would be superfluous to reproduce them here. We may mention, however, that Carleman's theorem shews that; if $f(z)$ is of order $\rho\left(>\frac{1}{2}\right)$, then its inverse function $\Phi(w)$ cannot have as many as 5ρ transcendant singularities in the finite part of the plane.

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